AE 211
STRENGTH OF MATERIALS

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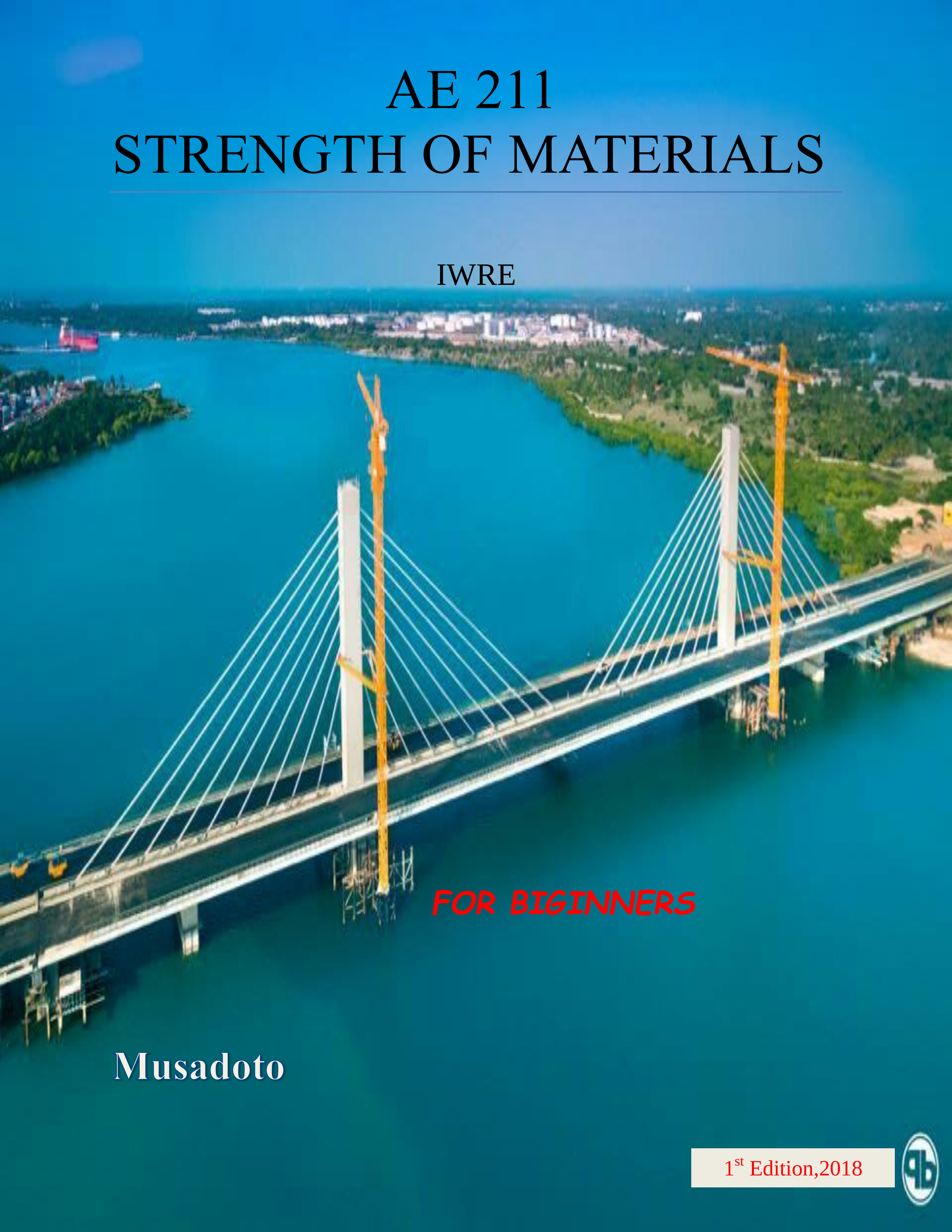


FOR BIGINNERS

Musadoto

1st Edition,2018

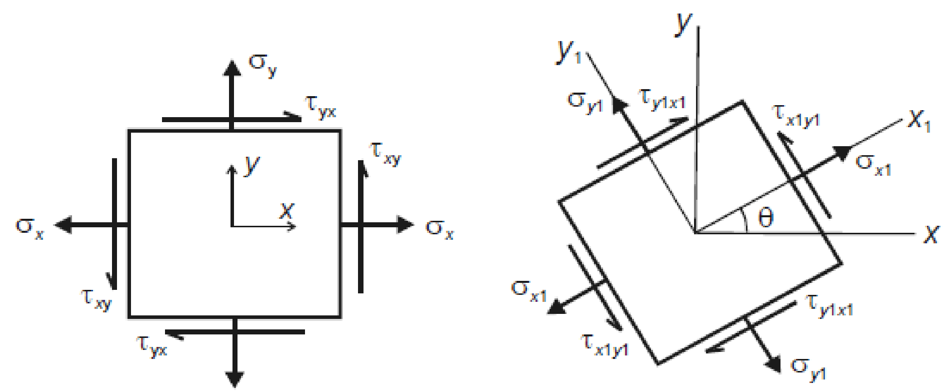
1

MOHR'S CIRCLE

Introduction

This is a **graphical** form of representing **Transformation equations** for plane stress.Named after **Christian Otto Mohr**, (8 October 1835 – 2 October 1918) was a German civil engineer.

Stress Transformation Equations



$$\boxed{\begin{aligned}
\sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\
\sigma_{y'} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\
\tau_{x'y'} &= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta
\end{aligned}}$$

If we vary θ from 0° to 360°, we will get all possible values of σx1 and τ_{x1y1} for a given stress state. It would be useful to represent σ_{x1} and τ_{x1y1} as functions of θ in graphical form. To do this, we must re-write the transformation equations:

$$\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \; + \; \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta \; + \; \tau_{xy}\cos 2\theta$$

Eliminate θ by squaring both sides of each equation and adding the two equations together.

$$\left(\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x1y1}^{\;2} = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}$$

Define $\boldsymbol{\sigma_{avg}}$ and ***R***

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} \qquad\qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}}$$

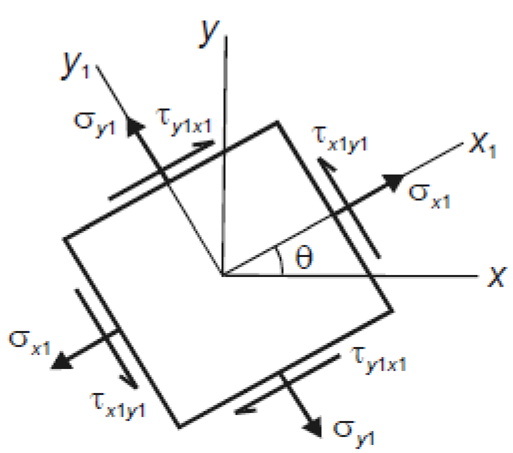
Substitute for $\boldsymbol{\sigma_{avg}}$ and ***R*** to get

$$\left(\sigma_{x1} - \sigma_{avg}\right)^2 + \tau_{x1y1}^{\;2} = R^2$$

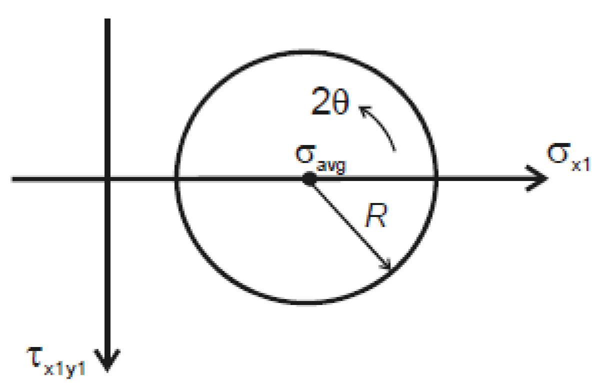
which is the equation for a **circle** with centre (σ_{avg},0) and radius *R*.

This circle is usually referred to as **Mohr's circle**, after the German civil engineer Otto Mohr (1835-1918). He developed the graphical technique for drawing the circle in 1882. The construction of Mohr's circle is one of the few graphical techniques still used in engineering. It provides a **simple** and **clear picture** of an otherwise complicated analysis.

Sign Convention for Mohr's Circle



$$\left(\sigma_{x1} - \sigma_{avg}\right)^2 + \tau_{x1y1}^{\;2} = R^2$$



Notice that shear stress is plotted as positive downward.

The reason for doing this is that 2θ is then positive counterclockwise, which agrees with the direction of 2θ used in the derivation of the tranformation equations and the direction of θ on the stress element.

Notice that although 2θ appears in Mohr's circle, θ appears on the stress element.

PROCEDURE FOR CONSTRUCTING MOHR'S CIRCLE

1. Draw a set of coordinate axes with σ_{x1} as abscissa (positive to the right) and τ_{x1y1} as ordinate (positive downward).

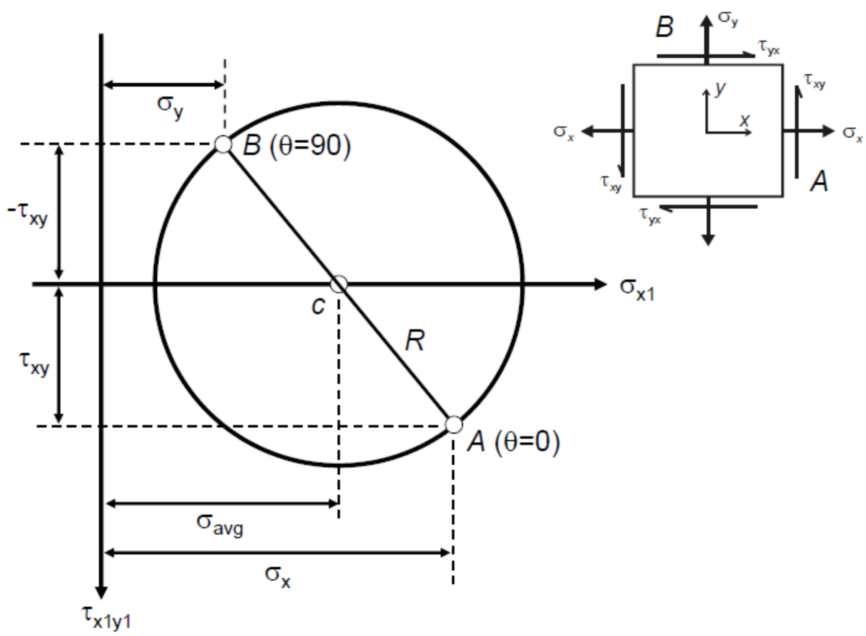
2. Locate the centre of the circle ***c*** at the point having coordinates $\sigma_{x1} = \sigma_{avg}$ and $\tau_{x1y1} = 0$.

3. Locate point ***A***, representing the stress conditions on the ***x*** face of the element by plotting its coordinates σx1 = σ_x and $\tau_{x1y1} = \tau_{xy}$. Note that point ***A*** on the circle corresponds to $\theta = 0°$.

4. Locate point *B*, representing the stress conditions on the *y* face of the element by plotting its coordinates $\sigma_{x1} = \sigma_y$ and $\tau_{x1y1} = -\tau_{xy}$. Note that point ***B*** on the circle corresponds to $\theta = 90°$.

5. Draw a line from point *A* to point *B*, a diameter of the circle passing through point *c*. Points *A* and *B* (representing stresses on planes at 90° to each other) are at opposite ends of the diameter (and therefore 180° apart on the circle).

6. Using point *c* as the centre, draw Mohr's circle through points *A* and *B*. This circle has radius *R*.



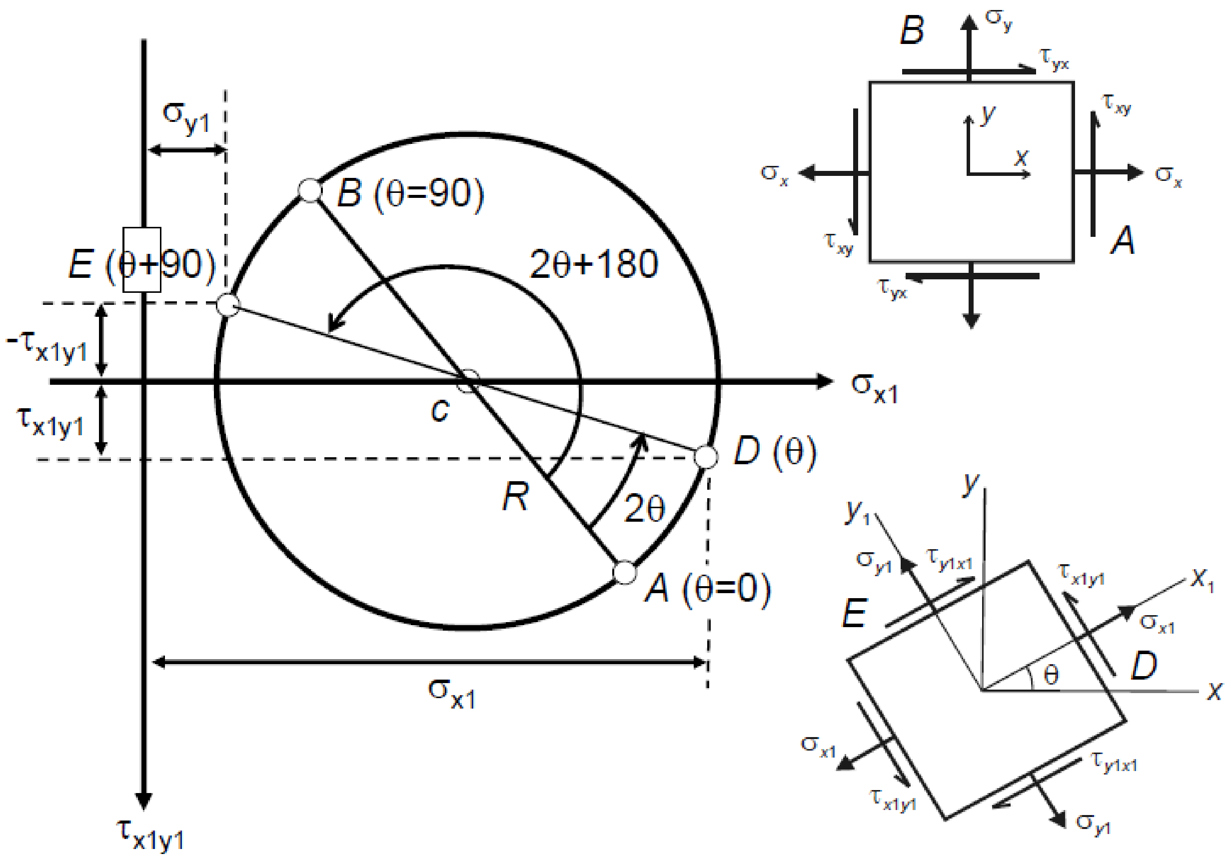
Stresses on an Inclined Element

1. On Mohr's circle, measure an angle 2θ counterclockwise from radius *cA*, because point *A* corresponds to θ = 0 and hence is the reference point from which angles are measured.

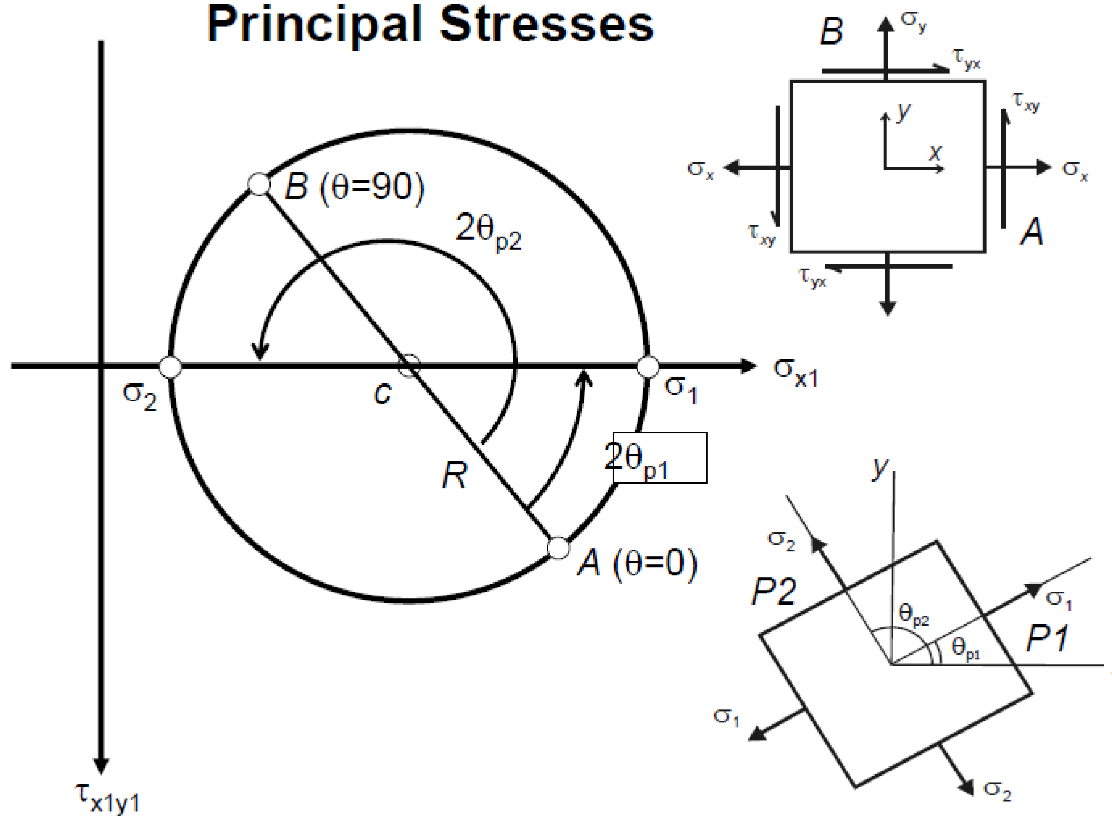
2. The angle 2θ locates the point ***D*** on the circle, which has coordinates σx1 and τ_{x1y1}. Point *D* represents the stresses on the *x1* face of the inclined element.

3. Point *E*, which is diametrically opposite point ***D*** on the circle, is located at an angle 2θ + 180° from ***cA*** (and 180° from ***cD***). Thus point ***E*** gives the stress on the *y1* face of the inclined element.

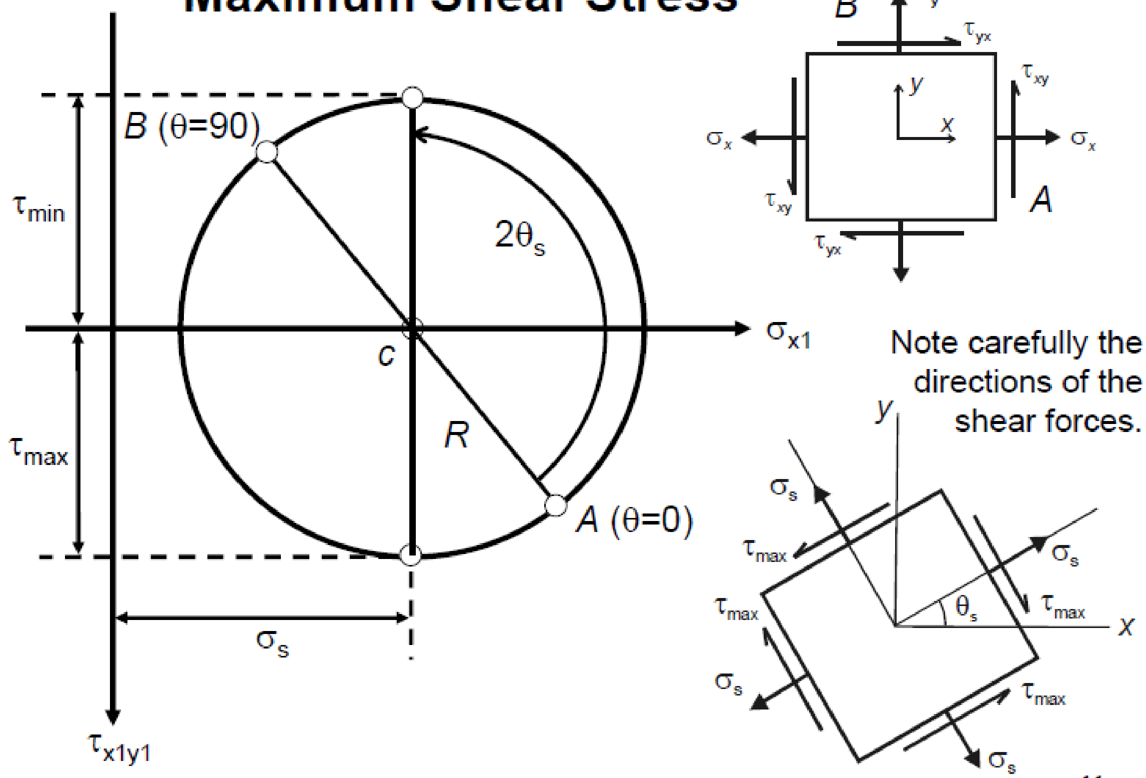
4. So, as we rotate the *x1y1* axes counterclockwise by an angle θ, the point on Mohr's circle corresponding to the *x1* face moves counterclockwise through an angle 2θ.



Principal Stresses



Maximum Shear Stress



Example:01

The state of plane stress at a point is represented by the stress element below. Draw the Mohr's circle, determine the principal stresses and the maximum shear stresses, and draw the corresponding stress elements.

$$c = \sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{-80+50}{2} = -15$$

$$R = \sqrt{(50-(-15))^2 + (25)^2}$$

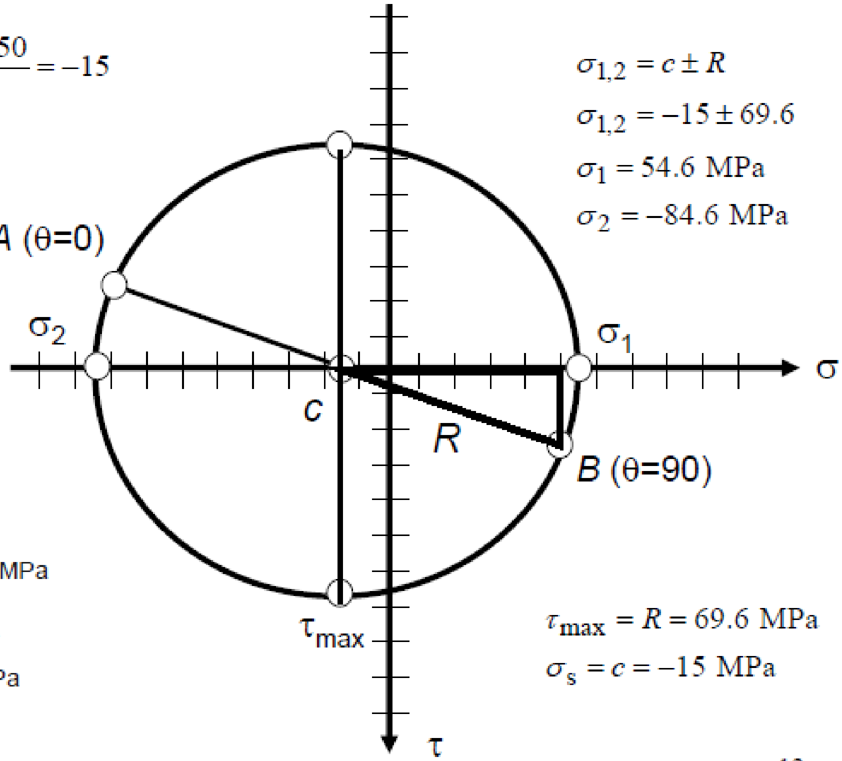
$$R = \sqrt{65^2 + 25^2} = 69.6$$

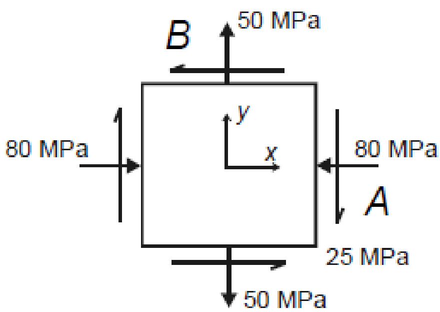
$$\sigma_{1,2} = c \pm R$$

$$\sigma_{1,2} = -15 \pm 69.6$$

$$\sigma_1 = 54.6 \text{ MPa}$$

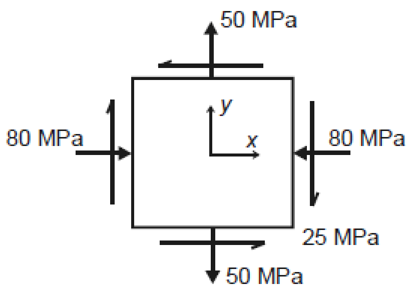
$$\sigma_2 = -84.6 \text{ MPa}$$





$$\tau_{max} = R = 69.6 \text{ MPa}$$

$$\sigma_s = c = -15 \text{ MPa}$$

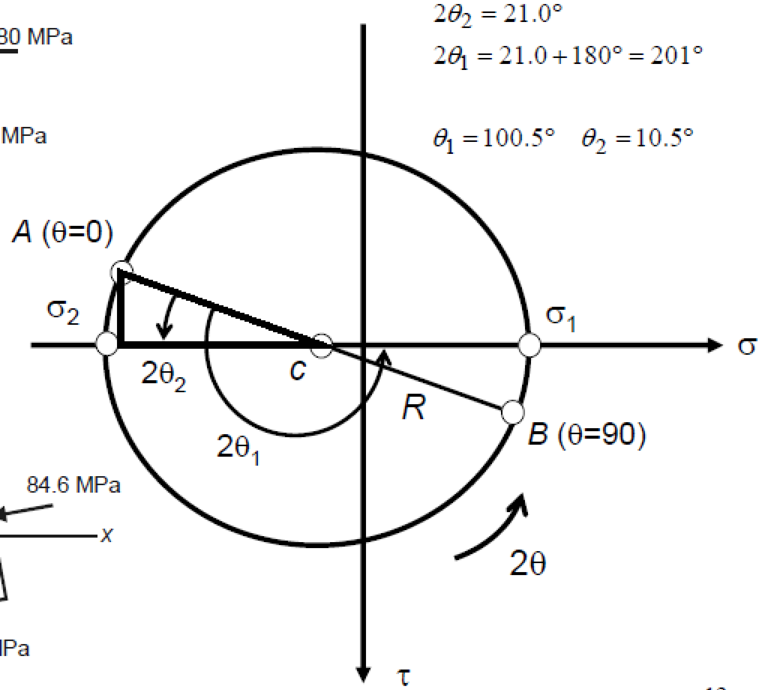


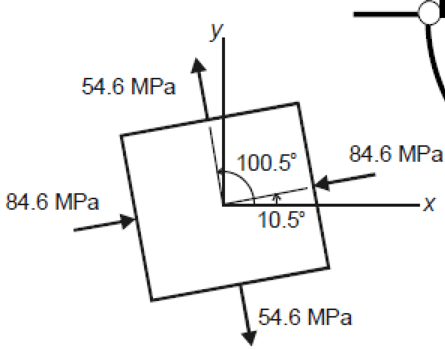
$$\tan 2\theta_2 = \frac{25}{80-15} = 0.3846$$

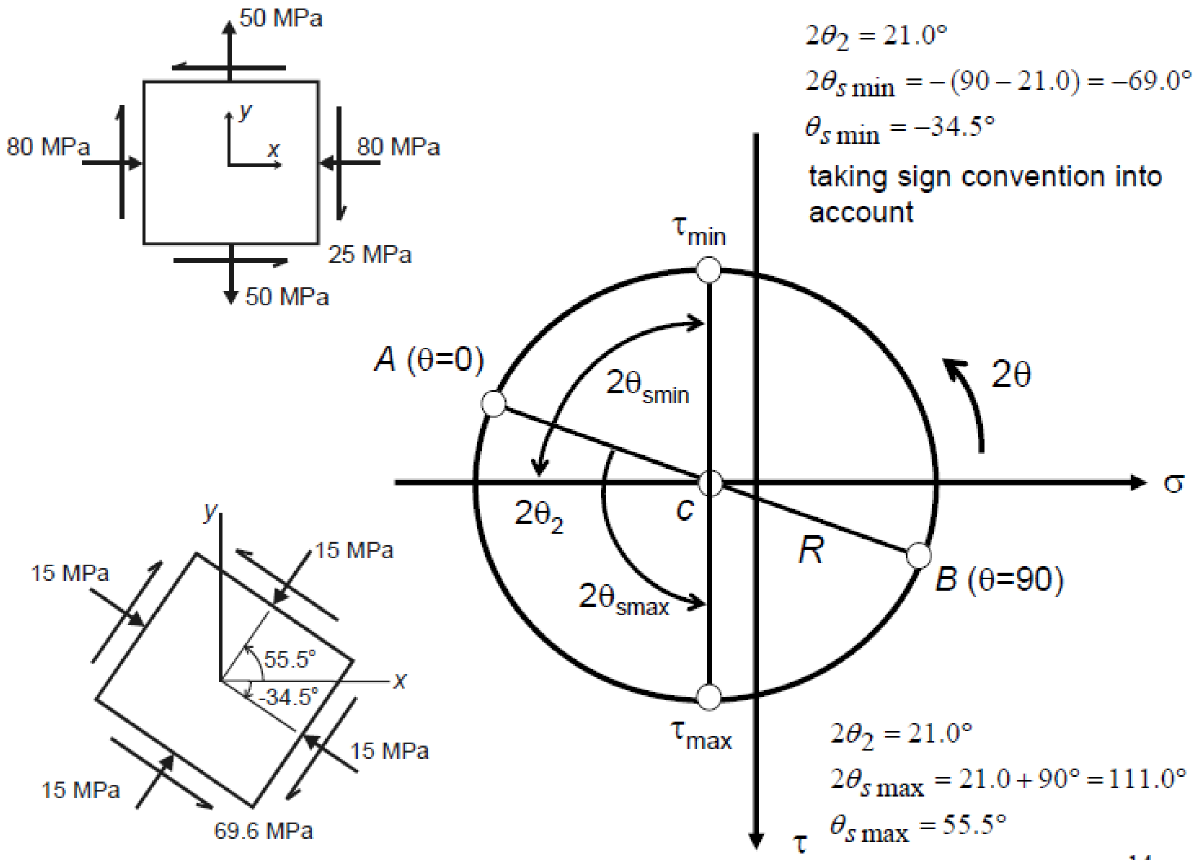
$$2\theta_2 = 21.0°$$

$$2\theta_1 = 21.0 + 180° = 201°$$

$$\theta_1 = 100.5° \quad \theta_2 = 10.5°$$

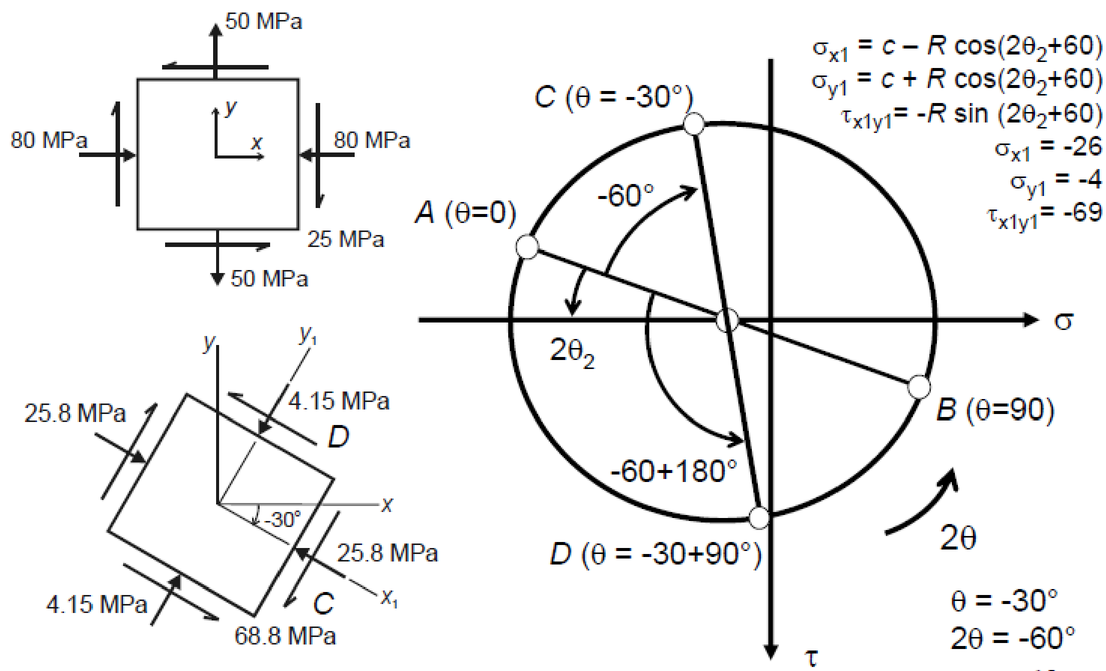






Example: 02

The state of plane stress at a point is represented by the stress element below. Find the stresses on an element inclined at 30° clockwise and draw the corresponding stress elements.



Principal Stresses σ_1 = 54.6 MPa, σ_2 = -84.6 MPa

But we have forgotten about the third principal stress!

Since the element is in plane stress (σ_z = 0), the third principal stress is zero.

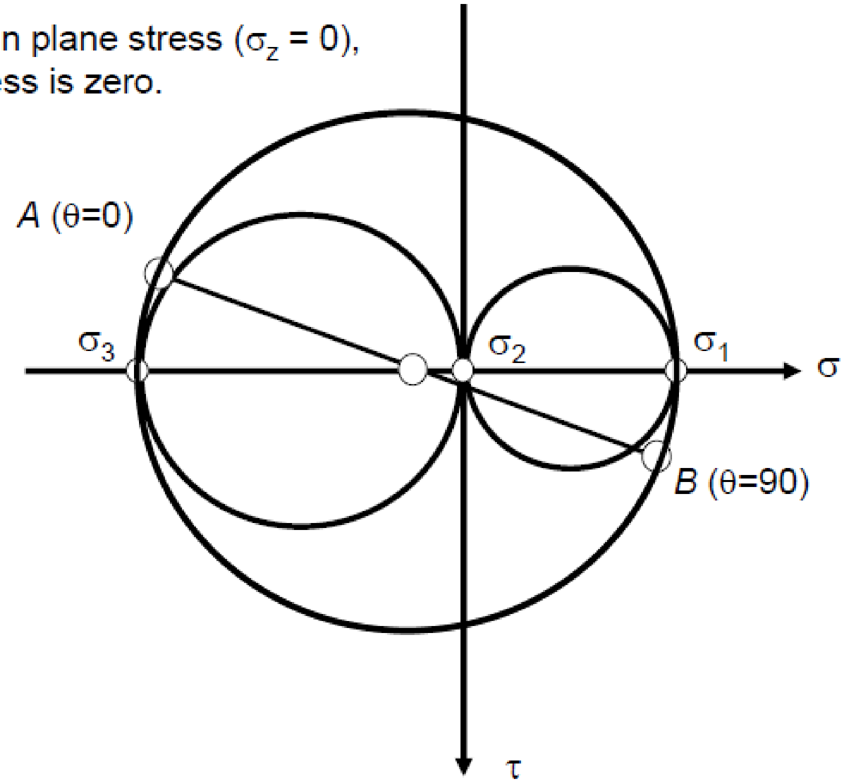
σ_1 = 54.6 MPa
σ_2 = 0 MPa
σ_3 = -84.6 MPa

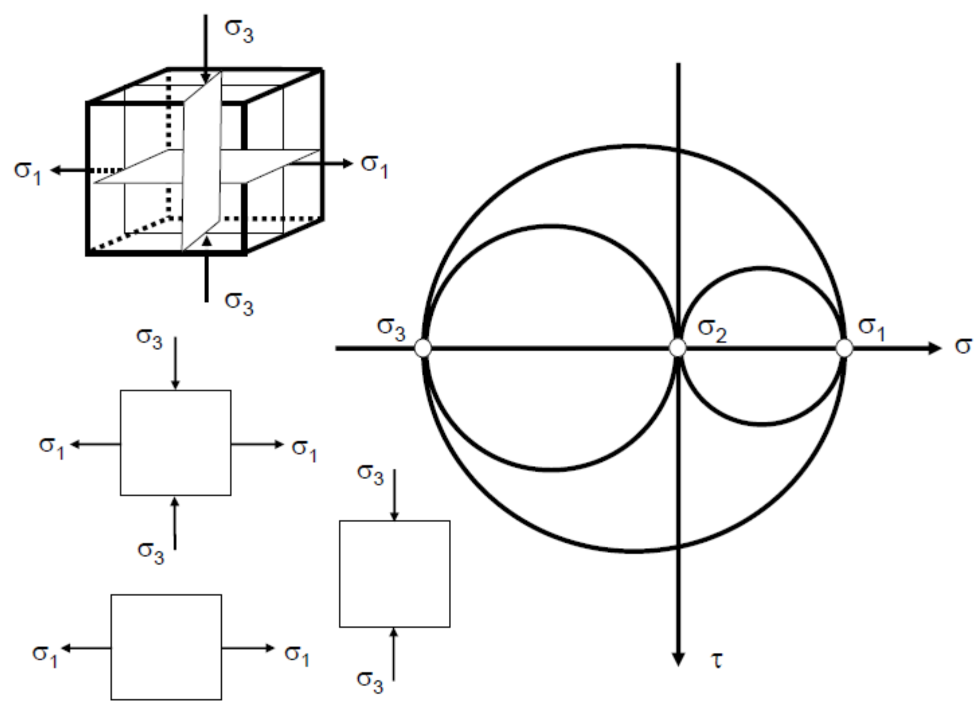
This means three Mohr's circles can be drawn, each based on two principal stresses:

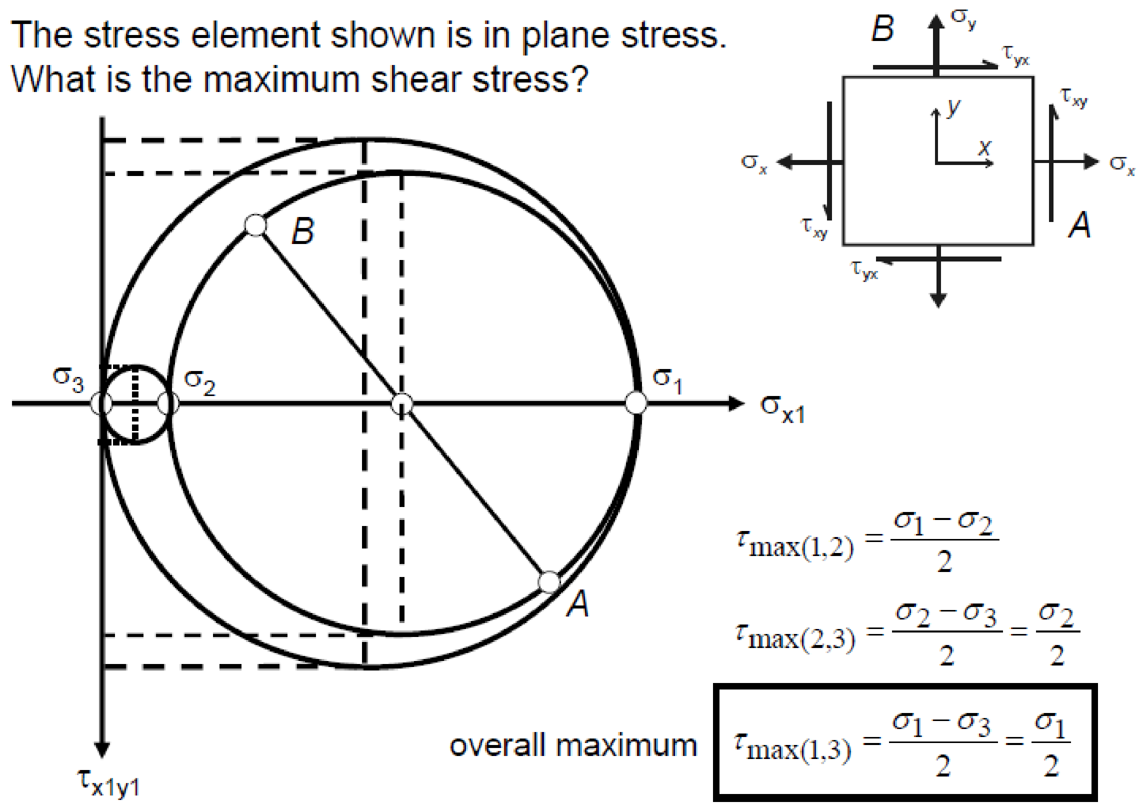
σ_1 and σ_3

σ_1 and σ_2

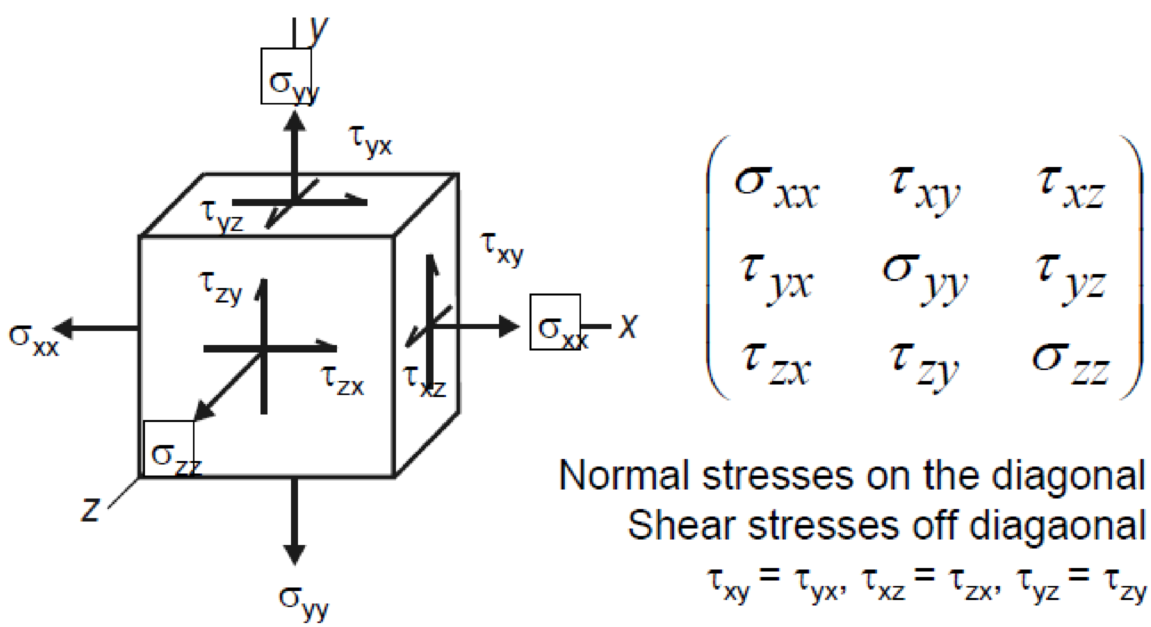
σ_2 and σ_3







INTRODUCTION TO THE STRESS TENSOR



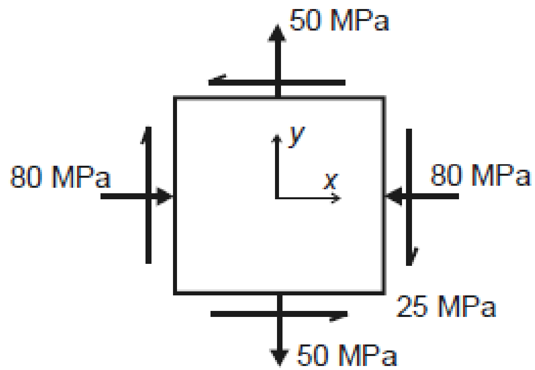
The normal and shear stresses on a stress element in 3D can be assembled into a 3x3 matrix known as the **STRESS TENSOR**.

From our analyses so far, we know that for a given stress system, it is possible to find a set of **three principal stresses**. We also know that if the principal stresses are acting, the shear stresses must be **zero**.
In terms of the stress tensor,

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \Longrightarrow \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

In mathematical terms:
This is the process of **matrix diagonalization** in which the **eigenvalues** of the original matrix are just the **principal stresses**.

Example:03
The state of plane stress at a point is represented by the stress element below. Find the principal stresses.



$$M = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} = \begin{pmatrix} -80 & -25 \\ -25 & 50 \end{pmatrix}$$

We must find the eigenvalues of this matrix. Remember the general idea of eigenvalues. We are looking for values of λ such that: ***A*r = λr** where **r** is a vector, and *A* is a matrix.
A*r – λr** = **0** or (A* – λ*I*)** **r** = **0** where *I* is the identity matrix.

For this equation to be true, either **r** = **0** or det **(*A* – λ*I*) = 0**.
Solving the latter equation (the "characteristic equation") gives us the eigenvalues λ_1 and λ_2.

$$\det\begin{pmatrix} -80-\lambda & -25 \\ -25 & 50-\lambda \end{pmatrix} = 0 \qquad \lambda^2 + 30\lambda - 4625 = 0$$

$$(-80-\lambda)(50-\lambda) - (-25)(-25) = 0 \quad \lambda = -84.6,\ 54.6$$

So, the principal stresses are –84.6 MPa and 54.6 MPa, as before. Knowing the eigenvalues, we can find the eigenvectors. These can be used to find the angles at which the principal stresses act. To find the eigenvectors, we substitute the eigenvalues into the equation **(*A* – λ*I*) r = 0** one at a time and solve for **r**.

$$\begin{pmatrix} -80-\lambda & -25 \\ -25 & 50-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -80-54.6 & -25 \\ -25 & 50-54.6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -134.6 & -25 \\ -25 & -4.64 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x = -0.186y$$

$\begin{pmatrix} -0.186 \\ 1 \end{pmatrix}$ is one eigenvector.

$$\begin{pmatrix} -80-\lambda & -25 \\ -25 & 50-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -80-(-84.6) & -25 \\ -25 & 50-(-84.6) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4.6 & -25 \\ -25 & 134.6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x = 5.388y$$

$\begin{pmatrix} 5.388 \\ 1 \end{pmatrix}$ is the other eigenvector.

Before finding the angles at which the principal stresses act, we can check to see if the eigenvectors are correct.

$$D = \begin{pmatrix} 54.6 & 0 \\ 0 & -84.6 \end{pmatrix} \quad C = \begin{pmatrix} -0.186 & 5.388 \\ 1 & 1 \end{pmatrix} \quad M = \begin{pmatrix} -80 & -25 \\ -25 & 50 \end{pmatrix}$$

$$D = C^{-1} M\, C$$

$$C^{-1} = \frac{1}{\det C} A^T \qquad \text{where } A = \text{matrix of co-factors}$$

$$C^{-1} = \begin{pmatrix} -0.179 & 0.967 \\ 0.179 & 0.033 \end{pmatrix}$$

$$D = \begin{pmatrix} -0.179 & 0.967 \\ 0.179 & 0.033 \end{pmatrix}\begin{pmatrix} -80 & -25 \\ -25 & 50 \end{pmatrix}\begin{pmatrix} -0.186 & 5.388 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 54.6 & 0 \\ 0 & -84.6 \end{pmatrix}$$

To find the angles, we must calculate the **unit** eigenvectors

$$\begin{pmatrix} -0.186 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -0.183 \\ 0.983 \end{pmatrix} \qquad \begin{pmatrix} 5.388 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0.938 \\ 0.183 \end{pmatrix}$$

And then assemble them into a rotation matrix *R* so that det *R* = +1.

$$R = \begin{pmatrix} 0.983 & -0.183 \\ 0.183 & 0.983 \end{pmatrix} \qquad \det R = (0.983)(0.983) - (0.183)(-0.183) = 1$$

The rotation matrix has the form

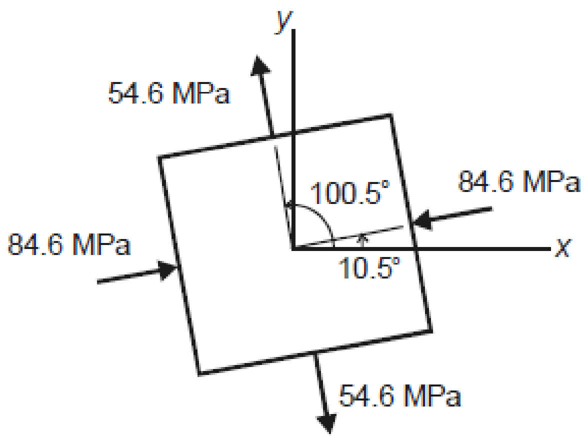
$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad D' = R^T M\ R$$

So θ = 10.5°, as we found earlier for one of the principal angles. Using the rotation angle of 10.5°, the matrix *M* (representing the original stress state of the element) can be transformed to matrix *D'* (representing the principal stress state).

$$D' = R^T M\ R$$

$$D' = \begin{pmatrix} 0.983 & 0.183 \\ -0.183 & 0.983 \end{pmatrix} \begin{pmatrix} -80 & -25 \\ -25 & 50 \end{pmatrix} \begin{pmatrix} 0.983 & -0.183 \\ 0.183 & 0.983 \end{pmatrix}$$

$$D' = \begin{pmatrix} -84.6 & 0 \\ 0 & 54.6 \end{pmatrix}$$



So, the transformation equations, Mohr's circle, and eigenvectors all give the same result for the **principal stress** element.

Finally, we can use the rotation matrix approach to find the stresses on an inclined element with θ = -30°.

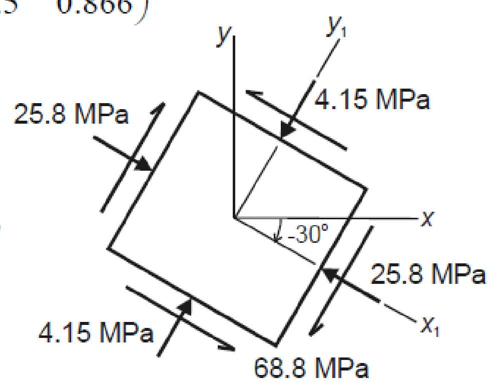
$$R = \begin{pmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{pmatrix} = \begin{pmatrix} 0.866 & 0.5 \\ 0.5 & 0.866 \end{pmatrix}$$

$$M' = R^T M \; R$$

$$M' = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix} \begin{pmatrix} -80 & -25 \\ -25 & 50 \end{pmatrix} \begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix}$$

$$M' = \begin{pmatrix} -25.8 & -68.8 \\ -68.8 & -4.15 \end{pmatrix} = \begin{pmatrix} \sigma_{x1} & \tau_{xy} \\ \tau_{yx} & \sigma_{y1} \end{pmatrix}$$

Again, the transformation equations, Mohr's circle, and the stress tensor approach all give the same result.



Example:04

A bar of cross section 850 mm^2 is acted upon by axial tensile forces of 60 kN applied at each end of the bar. Determine the normal and shearing stresses on a plane inclined at 30° to the direction of loading.

SOLUTION:

the normal stress on a cross section perpendicular to the axis of the bar is

$$\sigma_x = \frac{P}{A} = \frac{60 \times 10^3}{850 \times 10^{-6}} = 70 \times 10^6 \text{ Pa} \qquad \text{or} \qquad 70.6 \text{ MPa}$$

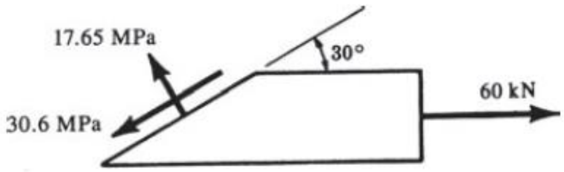
The normal stress on a plane at an angle Θ with the direction of loading

$$\sigma = \frac{1}{2}(70.6)(1 - \cos 60°) = 17.65 \text{ MPa}$$

The shearing stress on a plane at an angle q with the direction of loading

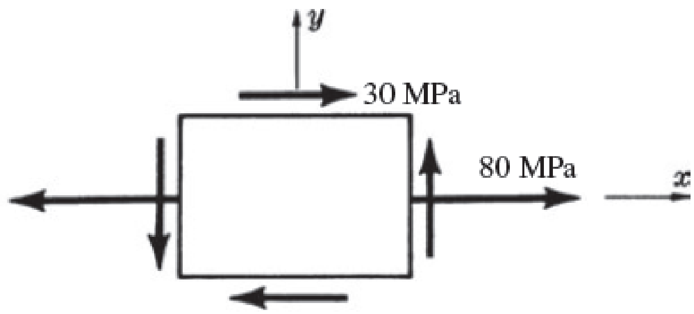
$$\tau = \frac{1}{2}(70.6)(\sin 60°) = 30.6 \text{ MPa}$$

These stresses together with the axial load of 60 kN are represented in Figure below



Example:05

A plane element in a body is subjected to a normal stress in the *x*-direction of 80 MPa, as well as a shearing stress of 30 MPa, as shown in Figure below. (*a*) Determine the normal and shearing stress intensities on a plane inclined at an angle of 30° to the normal stress. (*b*) Determine the maximum and minimum values of the normal stress that may exist on inclined planes and the directions of these stresses. (*c*) Determine the magnitude of the maximum shearing stress that may exist on an inclined plane.



(*a*) We have σ_x= 80 MPa and τ_{xy} = 30 MPa.The normal stress on a plane inclined at 30° to the *x*-axis is

$$\sigma = \frac{1}{2}\sigma_x - \frac{1}{2}\sigma_x \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$= \frac{1}{2}(80) - \frac{1}{2}(80)\cos 60° - 30 \sin 60° = -6 \text{ MPa}$$

The shearing stress on a plane inclined at 30° to the *x*-axis is

$$\tau = \frac{1}{2}\sigma_x \sin 2\theta - \tau_{xy} \cos 2\theta = \frac{1}{2}(80)\sin 60° - 30\cos 60° = 19.6 \text{ MPa}$$

(*b*) The values of the principal stresses, that is, the maximum and minimum values of the normal stresses existing in this element,

$$\sigma_{\max} = \frac{1}{2}\sigma_x + \sqrt{\left(\frac{1}{2}\sigma_x\right)^2 + (\tau_{xy})^2} = 40 + \sqrt{40^2 + 30^2} = 90 \text{ MPa}$$

$$\sigma_{\min} = \frac{1}{2}\sigma_x - \sqrt{\left(\frac{1}{2}\sigma_x\right)^2 + (\tau_{xy})^2} = 40 - \sqrt{40^2 + 30^2} = -10 \text{ MPa}$$

The directions of the planes on which these principal stresses occur

$$\tan 2\theta_p = \frac{\tau_{xy}}{\frac{1}{2}\sigma_x} = \frac{30}{40} \qquad \therefore 2\theta_p = 36.9°,\ 216.9°$$

Hence, $\theta_p = 18.4°,\ 108°$.

(c) The values of the maximum and minimum shearing stresses

$$\tau_{\substack{\max \\ \min}} = \pm\sqrt{\left(\frac{1}{2}\sigma_x\right)^2 + (\tau_{xy})^2} = \pm\sqrt{40^2 + 30^2} = \pm 50 \text{ MPa}$$

Example:06

Draw the Mohr's stress circle for direct stresses of 65MPa (tensile) and 35 MPa (compressive) and estimate the magnitude and direction of the resultant stresses on planes making angles of 20^0 and 65^0 with the plane of the first principal stress. Determine also the normal; and tangential stresses on these planes.

Solution

Given: $\sigma_x = +65 \text{ MN/m}^2$,

$\sigma_y = -35 \text{ MN/m}^2$, $\theta = 20°$ and $65°$

Analytical method:

Case 1:

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta$$

$$= \frac{65 + (-35)}{2} + \frac{65 - (-35)}{2}\cos(2 \times 20°)$$

$$= \mathbf{53.3\ MN/m^2}\ (\textit{tensile}) \quad \textbf{(Ans.)}$$

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta = \frac{65 - (-35)}{2}\sin(2 \times 20°)$$

$$= \mathbf{32.1\ MN/m^2}\ (\textit{shear}) \quad \textbf{(Ans.)}$$

$$\sigma_r = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(53.3)^2 + (32.1)^2}$$

$$= \mathbf{62.2\ MN/m^2} \quad \textbf{(Ans.)}$$

$$\tan\phi = \frac{\tau}{\sigma_n} = \frac{32.1}{53.3}$$

$$\boldsymbol{\phi = 31°} \quad \textbf{(Ans.)}$$

Case 2:

$$\sigma_n = \frac{65 + (-35)}{2} + \frac{65 - (-35)}{2} \cos(2 \times 65°) = -17.1 \text{ MN/m}^2$$

$$= \mathbf{17.1\ MN/m^2}\ (\textit{compressive}) \quad \text{(Ans.)}$$

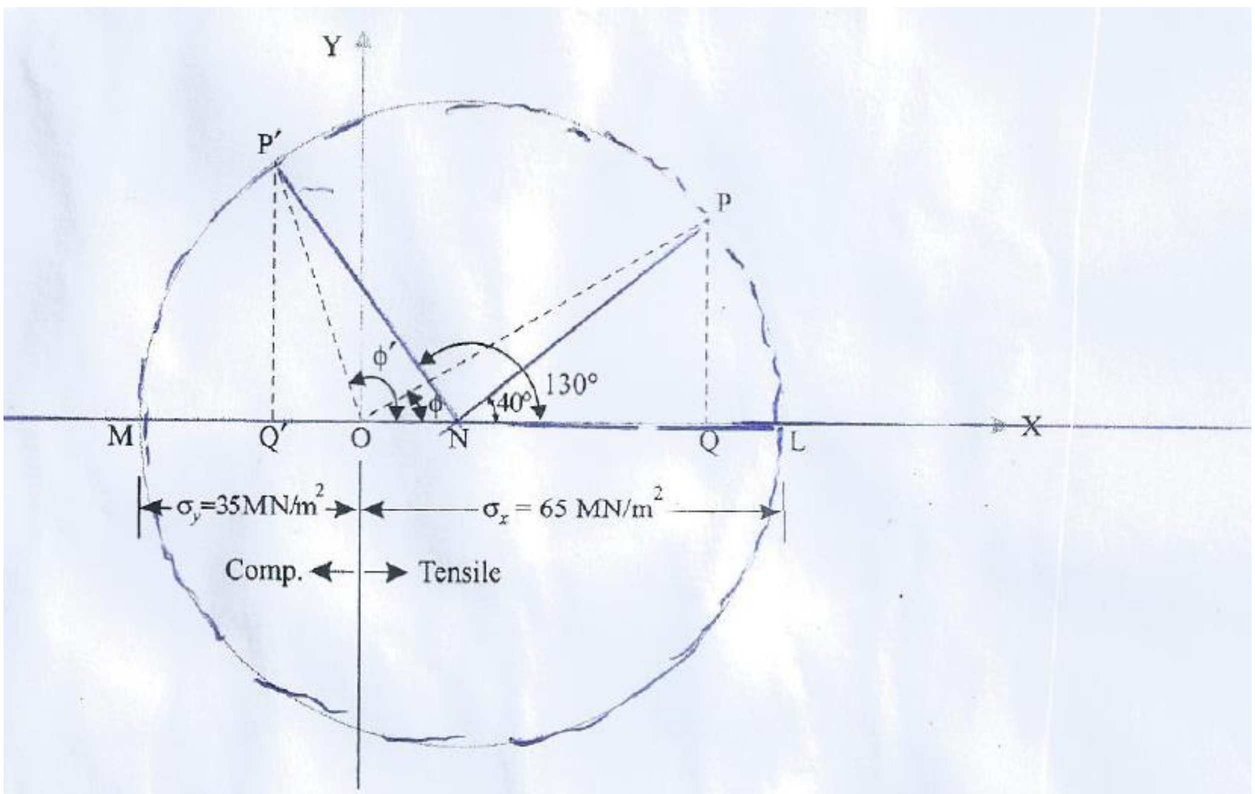
$$\tau = \frac{65 - (-35)}{2} \sin(2 \times 65°) = \mathbf{38.3}\ \text{MN/m}^2\ (\textit{shear}) \quad \textbf{(Ans.)}$$

$$\sigma_r = \sqrt{(17.1)^2 + (38.3)^2}$$

$$= \mathbf{41.9\ MN/m^2}\ \text{(Ans.)}$$

$$\phi = 180° - \tan^{-1}\frac{38.3}{17.1} = \mathbf{114°} \quad \text{(Ans.)}$$

GRAPHICALLY



Example :07

Show that in astrained material subjected to two dimensional stress, the sum of the normal components of stresses on any two mutually perpendicular planes is constant.

Solution. Let σ_x and σ_y be the principal stresses.

The normal stress (σ_n) on any plane at θ with the major principal plane,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

The normal stress (σ'_n) on the second plane at (θ + 90°) with the major principal plane,

$$\sigma'_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos (180° + 2\theta)$$

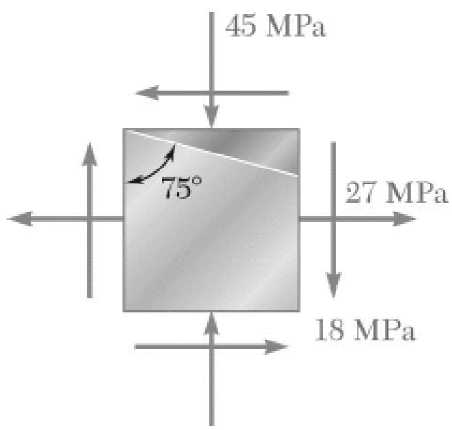
$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

$$\boldsymbol{\sigma_n + \sigma'_n = \sigma_x + \sigma_y = \text{constant}}$$

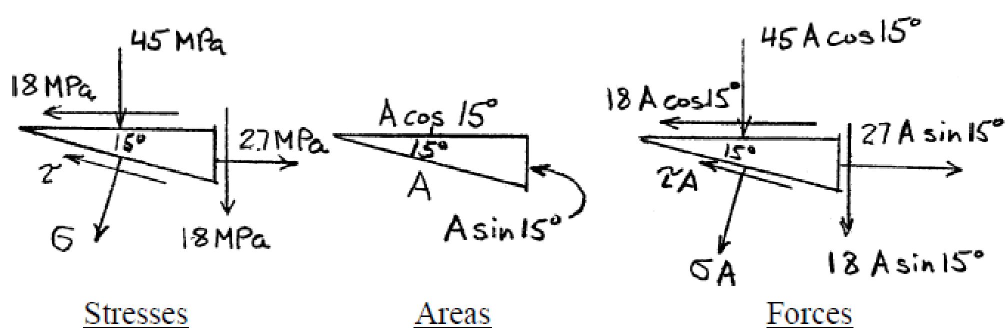
shown

Example:08

For the given state of stress, determine the normal and shearing stresses exerted on the oblique face of the shaded triangular element shown. Use a method of analysis based on the equilibrium of that element.



SOLUTION



$$\overset{+}{\swarrow}\ \Sigma F = 0: \quad \sigma A + 18A \cos 15° \sin 15°$$
$$+ 45A \cos 15° \cos 15° - 27A \sin 15° \sin 15°$$
$$+ 18A \sin 15° \cos 15° = 0$$

$$\sigma = -18\cos 15^\circ \sin 15^\circ - 45\cos^2 15^\circ$$
$$+27\sin^2 15^\circ - 18\sin 15^\circ \cos 15^\circ$$

$$\sigma = -49.2 \text{ MPa} \blacktriangleleft$$

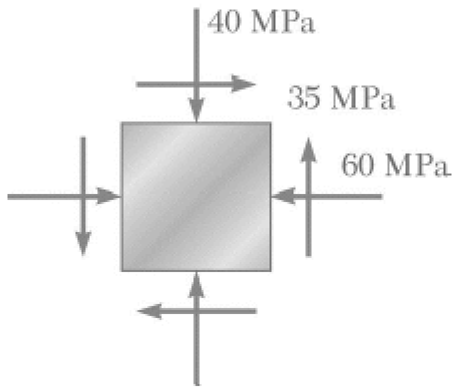
$$+\nwarrow \ \Sigma F = 0: \ \tau A + 18A\cos 15^\circ \cos 15^\circ$$
$$-45A\cos 15^\circ \sin 15^\circ$$
$$-27A\sin 15^\circ \cos 15^\circ$$
$$-18A\sin 15^\circ \sin 15^\circ = 0$$

$$\tau = -18(\cos^2 15^\circ - \sin^2 15^\circ) + (45+27)\cos 15^\circ \sin 15^\circ$$

$$\tau = 2.41 \text{ MPa} \blacktriangleleft$$

Example:09

For the given state of stress, determine (*a*) the principal planes, (*b*) the principal stresses.



SOLUTION

$$\sigma_x = -60 \text{ MPa} \quad \sigma_y = -40 \text{ MPa} \quad \tau_{xy} = 35 \text{ MPa}$$

(*a*) $$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{(2)(35)}{-60+40} = -3.50$$

$$2\theta_p = -74.05^\circ$$

$$\theta_p = -37.0^\circ, \quad 53.0^\circ \blacktriangleleft$$

(*b*) $$\sigma_{\text{max, min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-60-40}{2} \pm \sqrt{\left(\frac{-60+40}{2}\right)^2 + (35)^2}$$

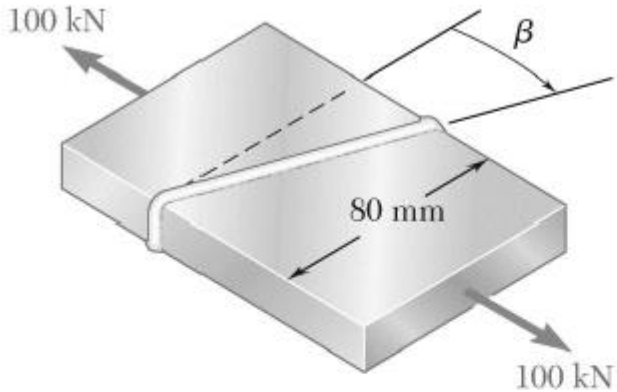
$$= -50 \pm 36.4 \text{ MPa}$$

$$\sigma_{\max} = -13.60 \text{ MPa} \blacktriangleleft$$

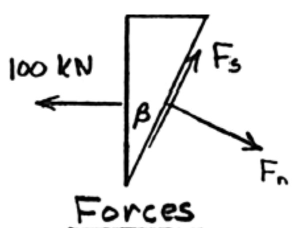
$$\sigma_{\min} = -86.4 \text{ MPa} \blacktriangleleft$$

Example :10

Two steel plates of uniform cross section 10 × 80 mm are welded together as shown. Knowing that centric 100-kN forces are applied to the welded plates and that the in-plane shearing stress parallel to the weld is 30 MPa, determine (*a*) the angle β, (*b*) the corresponding normal stress perpendicular to the weld.



SOLUTION



Area of weld:

$$A_w = \frac{(10\times10^{-3})(80\times10^{-3})}{\cos\beta} = \frac{800\times10^{-6}}{\cos\beta}\,\text{m}^2$$

(*a*) $\Sigma F_s = 0:\quad F_s - 100\sin\beta = 0 \qquad F_s = 100\sin\beta\ \text{kN} = 100\times10^3\sin\beta\ \text{N}$

$$\tau_w = \frac{F_s}{A_w} \qquad 30\times10^6 = \frac{100\times10^3\sin\beta}{800\times10^{-6}\ /\ \cos\beta} = 125\times10^6\sin\beta\cos\beta$$

$$\sin\beta\cos\beta = \frac{1}{2}\sin2\beta = \frac{30\times10^6}{125\times10^6} = 0.240 \qquad \beta = 14.34^\circ \blacktriangleleft$$

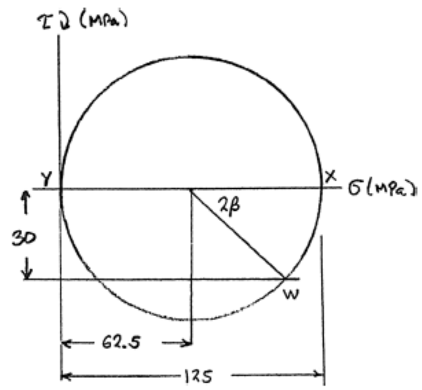
(*b*) $\Sigma F_n = 0:\quad F_n - 100\cos\beta = 0 \qquad F_n = 100\cos14.34^\circ = 96.88\ \text{kN}$

$$A_w = \frac{800\times10^{-6}}{\cos14.34} = 825.74\times10^{-6}\,\text{m}^2$$

$$\sigma = \frac{F_n}{A_w} = \frac{96.88\times10^3}{825.74\times10^{-6}} = 117.3\times10^6\ \text{Pa} \qquad \sigma = 117.3\ \text{MPa} \blacktriangleleft$$

Using mohr’s cycle

SOLUTION



$$\sigma_x = \frac{P}{A} = \frac{100\times10^3}{(10\times10^{-3})(80\times10^{-3})} = 125\times10^6\ \text{Pa} = 125\ \text{MPa}$$

$$\sigma_y = 0 \qquad \tau_{xy} = 0$$

From Mohr's circle:

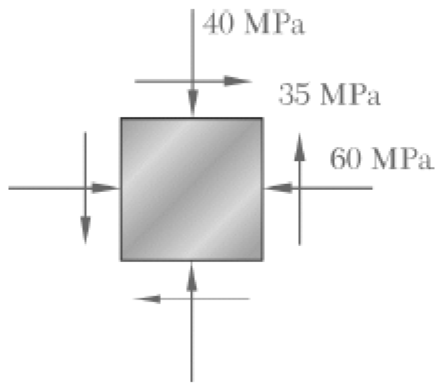
(*a*) $\sin 2\beta = \dfrac{30}{62.5} = 0.48$ $\qquad \beta = 14.3°$ ◀

(*b*) $\sigma = 62.5 + 62.5\cos 2\beta$

$\sigma = 117.3\ \text{MPa}$ ◀

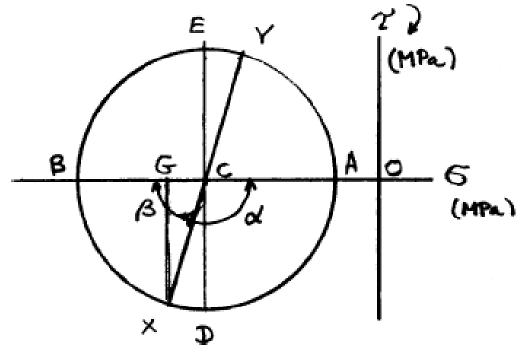
Example:11

For the given state of stress, determine (*a*) the orientation of the planes of maximum in-plane shearing stress, (*b*) the maximum in-plane shearing stress, (*c*) the corresponding normal stress



SOLUTION

$$\sigma_x = -60 \text{ MPa},$$
$$\sigma_y = -40 \text{ MPa},$$
$$\tau_{xy} = 35 \text{ MPa}$$
$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = -50 \text{ MPa}$$



Plotted points for Mohr's circle:

$$X: (\sigma_x, -\tau_{xy}) = (-60 \text{ MPa}, -35 \text{ MPa})$$
$$Y: (\sigma_y, \tau_{xy}) = (-40 \text{ MPa}, 35 \text{ MPa})$$
$$C: (\sigma_{\text{ave}}, 0) = (-50 \text{ MPa}, 0)$$

(*a*) $\tan\beta = \frac{GX}{CG} = \frac{35}{10} = 3.500$

$\beta = 74.05°$

$\theta_b = -\frac{1}{2}\beta = -37.03°$ $\qquad\qquad \theta_b = -37.0°$ ◀

$\alpha = 180° - \beta = 105.95°$

$\theta_a = \frac{1}{2}\alpha = 52.97°$ $\qquad\qquad \theta_a = 53.0°$ ◀

$R = \sqrt{\overline{CG}^2 + \overline{GX}^2} = \sqrt{10^2 + 35^2} = 36.4 \text{ MPa}$

(*b*) $\sigma_{\min} = \sigma_{\text{ave}} - R = -50 - 36.4$ $\qquad\qquad \sigma_{\min} = -86.4 \text{ MPa}$ ◀

$\sigma_{\max} = \sigma_{\text{ave}} + R = -50 + 36.4$ $\qquad\qquad \sigma_{\max} = -13.6 \text{ MPa}$ ◀

(*a′*) $\theta_d = \theta_B + 45° = 7.97°$ $\qquad\qquad \theta_d = 8.0°$ ◀

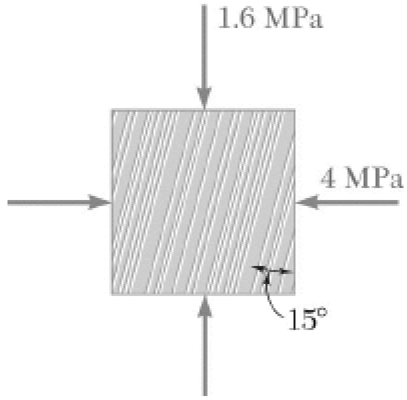
$\theta_e = \theta_A + 45° = 97.97°$ $\qquad\qquad \theta_e = 98.0°$ ◀

$\tau_{\max} = R = 36.4 \text{ MPa}$ $\qquad\qquad \tau_{\max} = 36.4 \text{ MPa}$ ◀

(*b′*) $\sigma' = \sigma_{\text{ave}} = -50 \text{ MPa}$ $\qquad\qquad \sigma' = -50.0 \text{ MPa}$ ◀

Example:12

The grain of a wooden member forms an angle of 15° with the vertical. For the state of stress shown, determine (*a*) the in-plane shearing stress parallel to the grain, (*b*) the normal stress perpendicular to the grain



SOLUTION

$$\sigma_x = -4 \text{ MPa} \quad \sigma_y = -1.6 \text{ MPa} \quad \tau_{xy} = 0$$

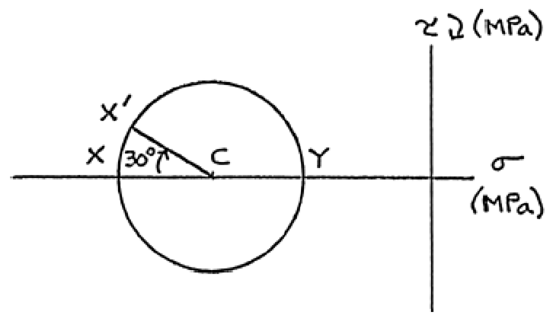
$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = -2.8 \text{ MPa}$$

Plotted points for Mohr's circle:

$$X: (\sigma_x, -\tau_{xy}) = (-4 \text{ MPa}, 0)$$

$$Y: (\sigma_y, \tau_{xy}) = (-1.6 \text{ MPa}, 0)$$

$$C: (\sigma_{\text{ave}}, 0) = (-2.8 \text{ MPa}, 0)$$



$\underline{\theta = -15^\circ}$. $\quad 2\theta = -30^\circ$

$\overline{CX} = 1.2 \text{ MPa} \quad R = 1.2 \text{ MPa}$

(*a*) $\quad \tau_{x'y'} = -\overline{CX'} \sin 30^\circ = -R \sin 30^\circ = -1.2 \sin 30^\circ \qquad \tau_{x'y'} = -0.600 \text{ MPa}$ ◀

(*b*) $\quad \sigma_{x'} = \sigma_{\text{ave}} - \overline{CX'} \cos 30^\circ = -2.8 - 1.2 \cos 30^\circ \qquad \sigma_{x'} = -3.84 \text{ MPa}$ ◀

Example :13

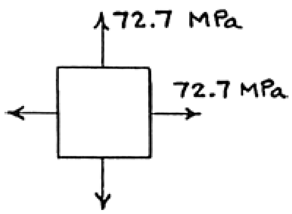
A spherical gas container made of steel has a 5-m outer diameter and a wall thickness of 6 mm. Knowing that the internal pressure is 350 kPa, determine the maximum normal stress and the maximum shearing stress in the container.

SOLUTION

$$d = 5\text{ m} \quad t = 6\text{ mm} = 0.006\text{ m}, \quad r = \frac{d}{2} - t = 2.494\text{ m}$$

$$\sigma = \frac{pr}{2t} = \frac{(350 \times 10^3\text{ Pa})(2.494\text{ m})}{2(0.006\text{ m})} = 72.742 \times 10^6\text{ Pa}$$

$\sigma = 72.7\text{ MPa}$ ◀



$$\sigma_{max} = 72.742\text{ MPa}$$

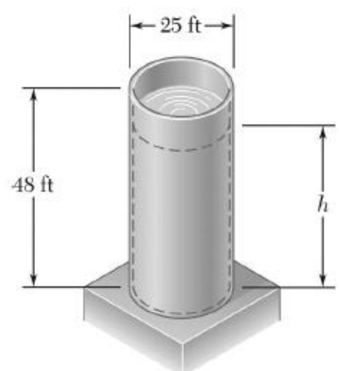
$$\sigma_{min} \approx 0 \quad \text{(Neglecting small radial stress)}$$

$$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min})$$

$\tau_{max} = 36.4\text{ MPa}$ ◀

Example :14 (English SI UNITS)

The unpressurized cylindrical storage tank shown has a $\frac{3}{16}$-in. wall thickness and is made of steel having a 60-ksi ultimate strength in tension. Determine the maximum height *h* to which it can be filled with water if a factor of safety of 4.0 is desired. (Specific weight of water = 62.4 lb/ft^3.) (refer test 1 2017/18 AE 211) for new concept of thicknes **t.**



SOLUTION

$$d_0 = (25)(12) = 300\text{ in.}$$

$$r = \frac{1}{2}d - t = 150 - \frac{3}{16} = 149.81\text{ in.}$$

$$\sigma_{all} = \frac{\sigma_U}{F.S.} = \frac{60\text{ ksi}}{4.0} = 15\text{ ksi} = 15 \times 10^3\text{psi}$$

$$\sigma_{all} = \frac{pr}{t}$$

$$p = \frac{t\sigma_{all}}{r} = \frac{\left(\frac{3}{16}\right)(15 \times 10^3)}{149.81} = 18.77\text{ psi} = 2703\text{ lb/ft}^2$$

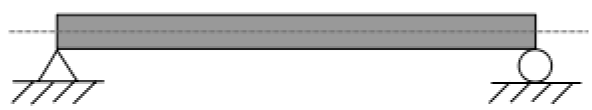
But $p = \gamma h$, $\quad h = \dfrac{p}{\gamma} = \dfrac{2703\text{ lb/ft}^2}{62.4\text{ lb/ft}^3}$ $\qquad h = 43.3\text{ ft}$ ◀

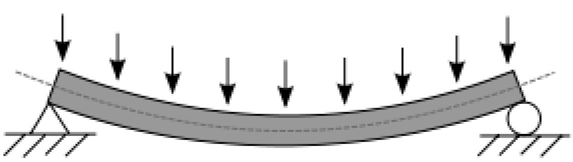
2

BEAMS

A **beam** is a structural element that primarily resists loads applied laterally to the beam's axis. Its mode of deflection is primarily by bending. The loads applied to the beam result in reaction forces at the beam's support points. The total effect of all the forces acting on the beam is to produce **shear forces** and **bending moments** within the beam, that in turn induce internal stresses, strains and deflections of the beam. Beams are characterized by their manner of support, profile (shape of cross-section), length, and their material.

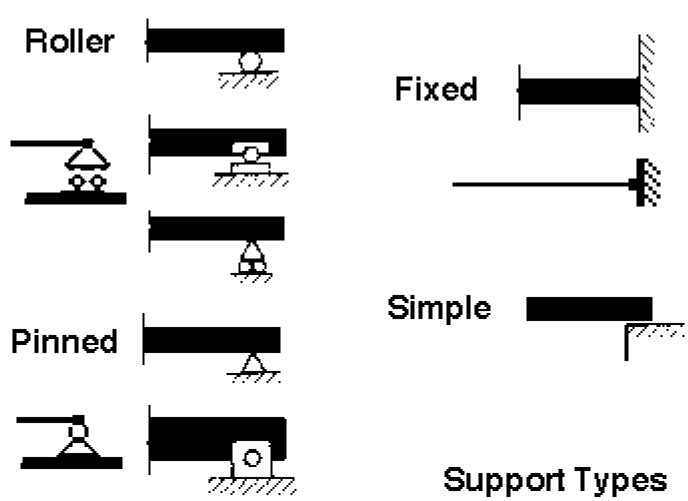
Beams are traditionally descriptions of building or civil engineering structural elements, but any structures such as automotive automobile frames, aircraft components, machine frames, and other mechanical or structural systems contain beam structures that are designed to carry lateral loads are analyzed in a similar fashion.





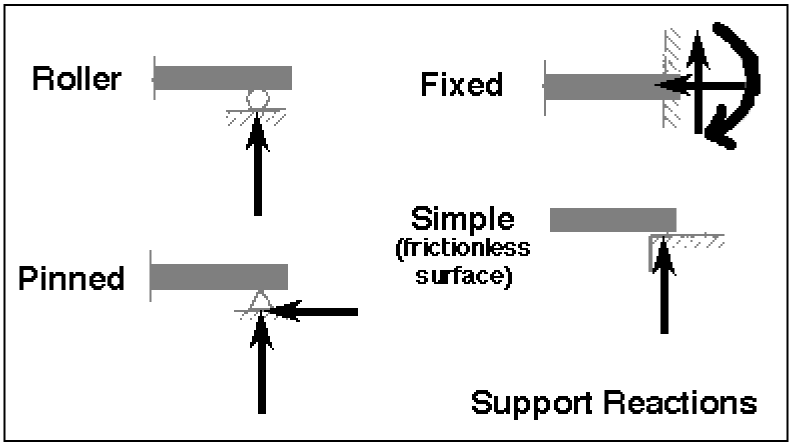
Types of beam supports

- Pinned support or Hinged support
- Roller support
- Fixed support



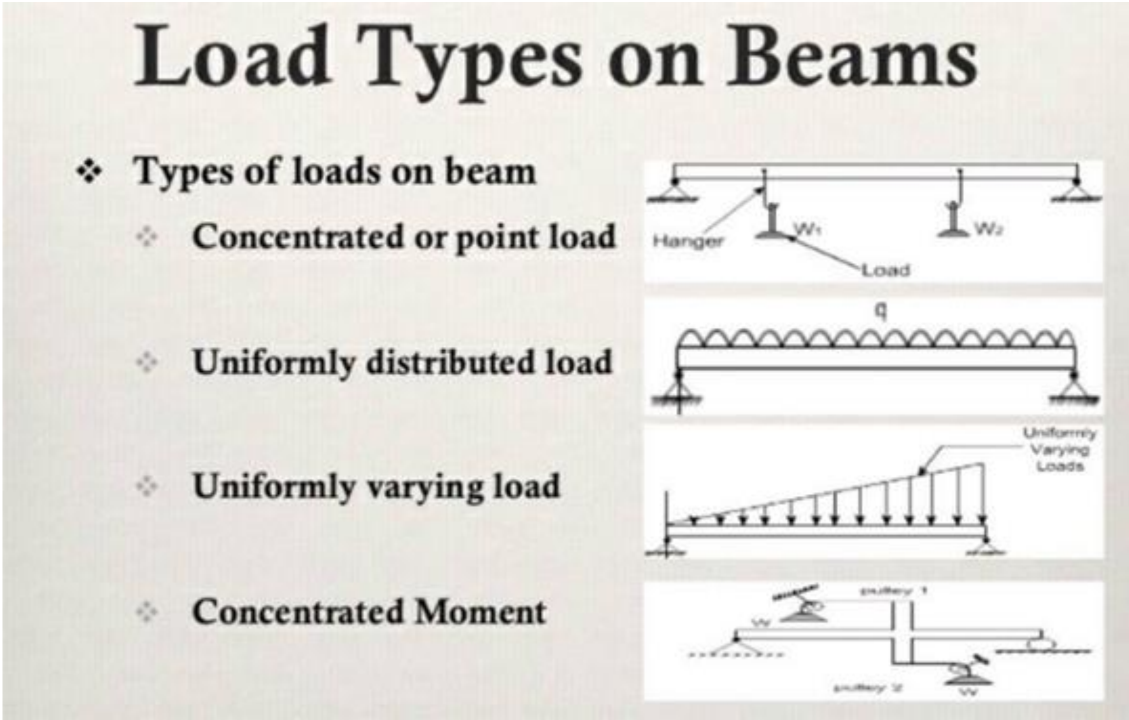
Types of beams

- Simply supported beam
- Cantilever beam
- Beam with overhang



Types of Loads

- Concentrated load
- Uniform distributed Load (UDL)
- Linearly Varying load
- Concentrated Moment



EVALUATION OF REACTIVE FORCES

A beam structure reaction can be calculated by determining its equilibrium such that:

$$\sum \mathbf{F} = 0 \qquad\qquad \sum \mathbf{M} = 0$$

$$\swarrow \quad \searrow \qquad\qquad \downarrow$$

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_z = 0$$

Lets make beam analysis by categorizing in the following groups

1. Shear Forces and Bemding Moment
2. Stresses in beams
3. Deflection of beams

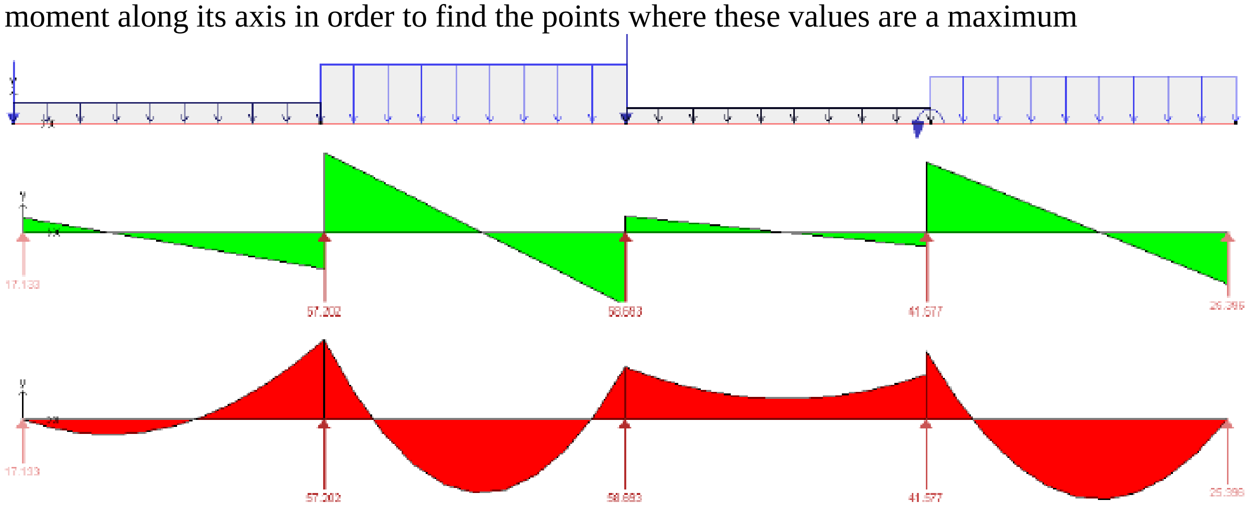
1. <u>SHEAR FORCES AND BEMDING MOMENT</u>

Types of beam

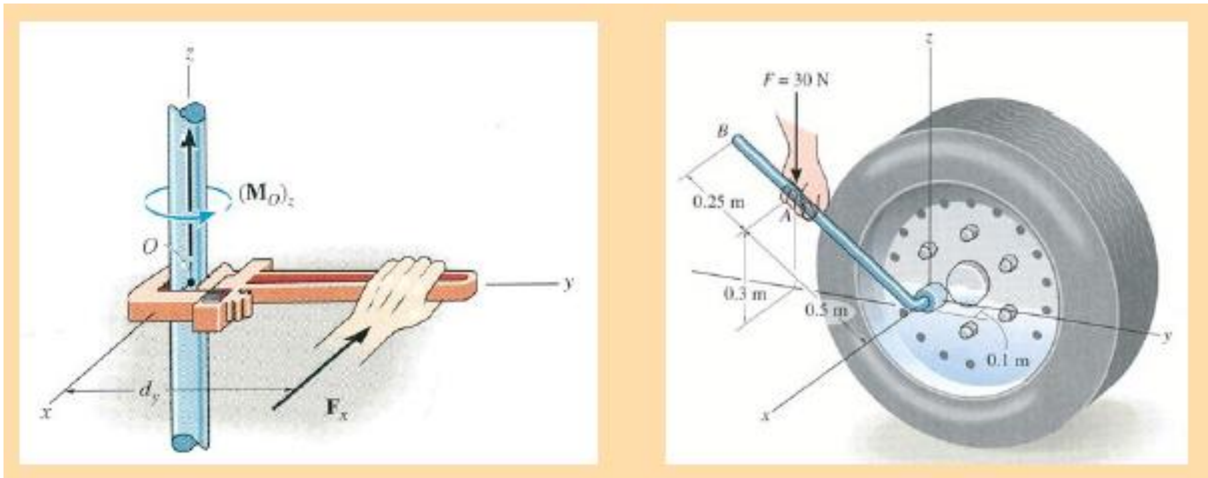
a) **Determinate Beam**
   The force and moment of reactions at supports can be determined by using the 3 equilibrium equations of statics i.e. ΣFx= 0, ΣFy= 0 and ΣM = 0

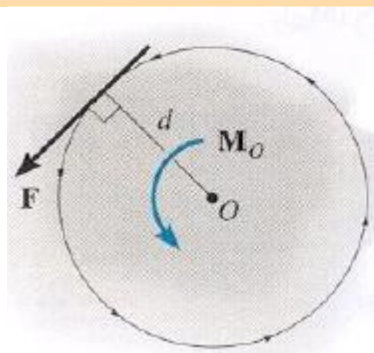
b) Indeterminate Beam
   The force and moment of reactions at supports are more than the number of equilibrium equations of statics. (The extra reactions are called redundant and represent the amount of degrees of indeterminacy).

In order to properly design a beam, it is important to know the variation of the shear and moment along its axis in order to find the points where these values are a maximum



PRINCIPLE OF MOMENTS

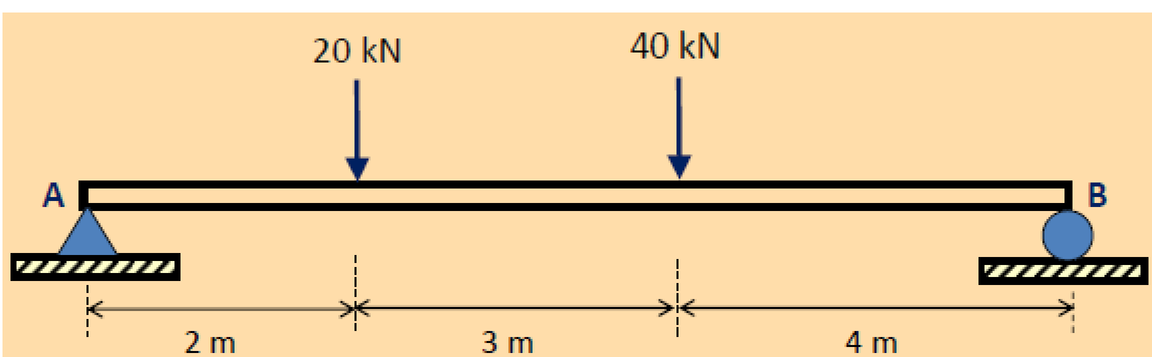




In the 2-D case, the magnitude of the moment is: ***Mo = Force x distance***

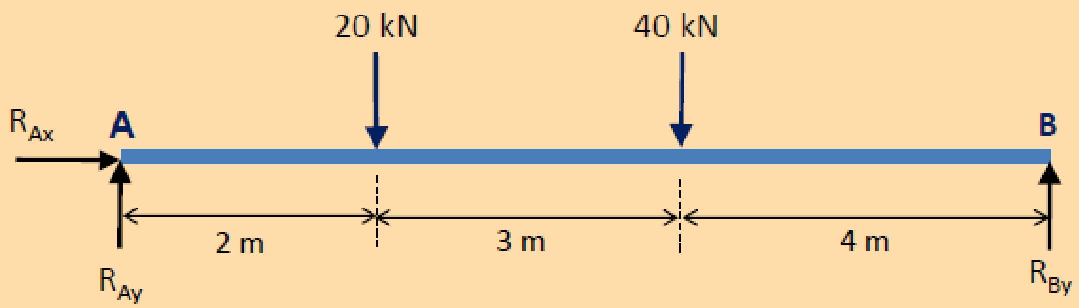
Example:01

The beam shown below is supported by a pin at A and roller at B. Calculate the reactions at both supports due to the loading.



SOLUTION

Draw the free body diagram:



By taking the moment at B,

$\Sigma M_B = 0$

$R_{Ay} \times 9 - 20 \times 7 - 40 \times 4 = 0$

$9R_{Ay} = 140 + 160$

$R_{Ay} = 33.3$ kN

$\Sigma F_y = 0$

$R_{Ay} + R_{By} - 20 - 40 = 0$

$R_{By} = 20 + 40 - 33.3$

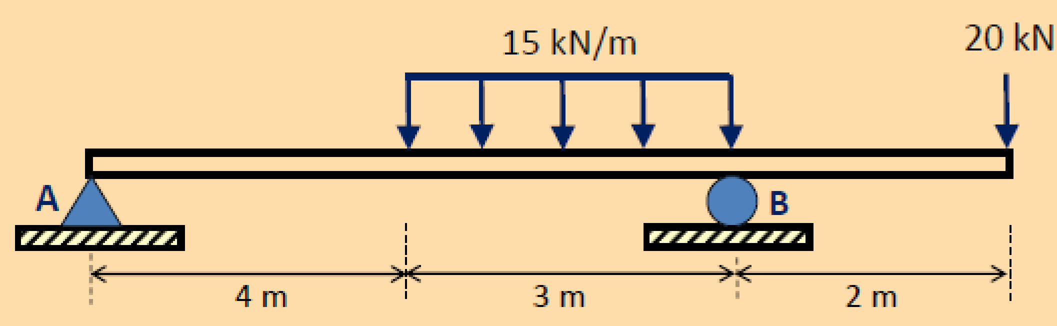
$R_{By} = 26.7$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$

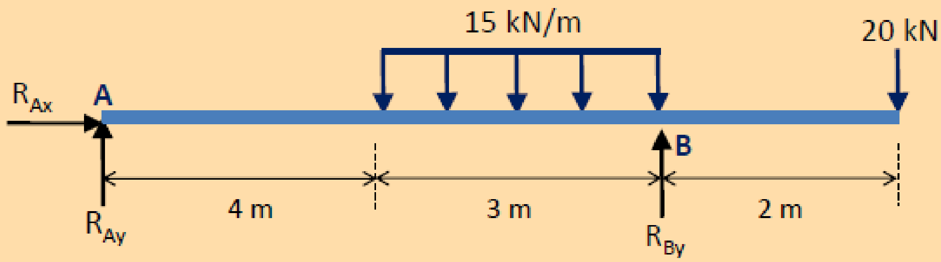
Example:02

Determine the reactions at support A and B for the overhanging beam subjected to the loading as shown.



SOLUTION

Draw the free body diagram:



By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times 7 + 20 \times 9 - (15 \times 3) \times 5.5 = 0$

$7R_{By} = 247.5 + 180$

$R_{By} = 61.07$ kN

$\Sigma F_y = 0$

$R_{Ay} + R_{By} - 20 - 45 = 0$

$R_{Ay} = 20 + 45 - 61.07$

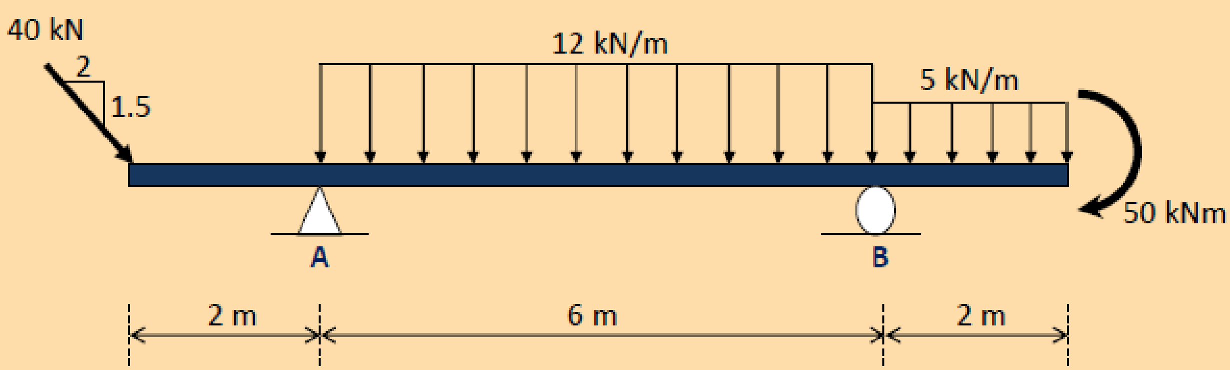
$R_{Ay} = 3.93$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$

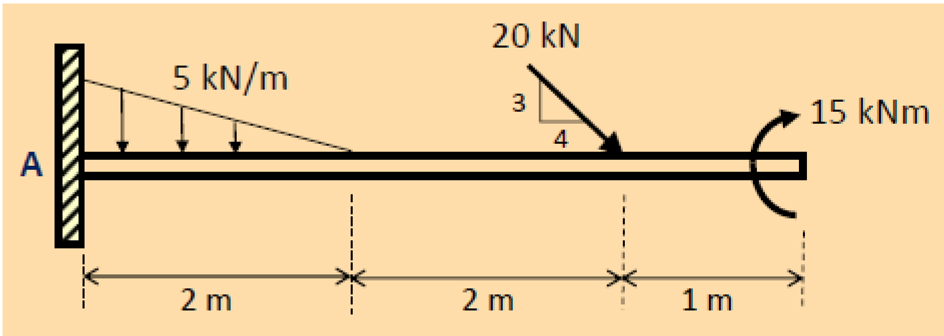
Example :03

Determine the reactions from figure shown below.



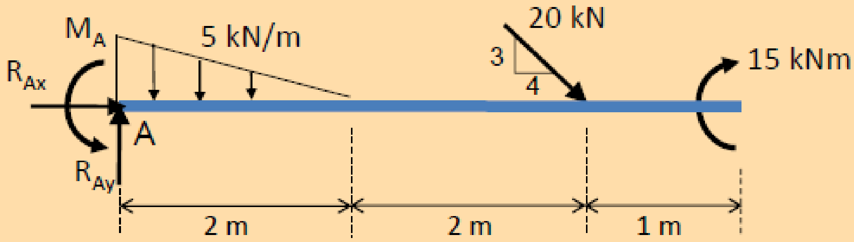
Example:04

A cantilever beam is loaded as shown. Determine all reactions at support A.



SOLUTION

Draw the free body diagram:



$\Sigma F_x = 0$

$-R_{Ax} + 20\,(4/5) = 0$

$-R_{Ax} = 16$ kN

$\Sigma F_y = 0$

$R_{Ay} - 0.5\,(5)(2) - 20(3/5) = 0$

$R_{Ay} - 5 - 12 = 0$

$R_{Ay} = 17$ kN

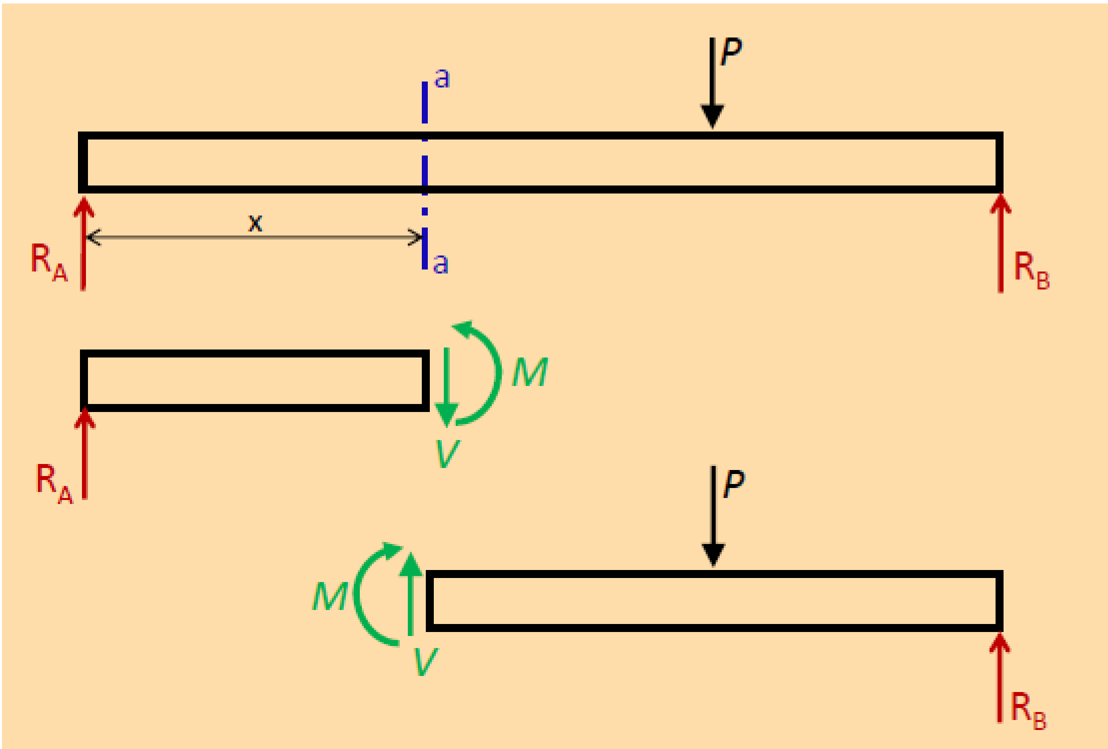
$\Sigma M_A = 0$

$-M_A + 0.5(5)(2)(1/3)(2) + 20(3/5)\,(4) + 15 = 0$

$M_A = 3.3 + 48 + 15$

$M_A = 66.3$ kNm

SHEAR FORCE & BENDING MOMENT DIAGRAM



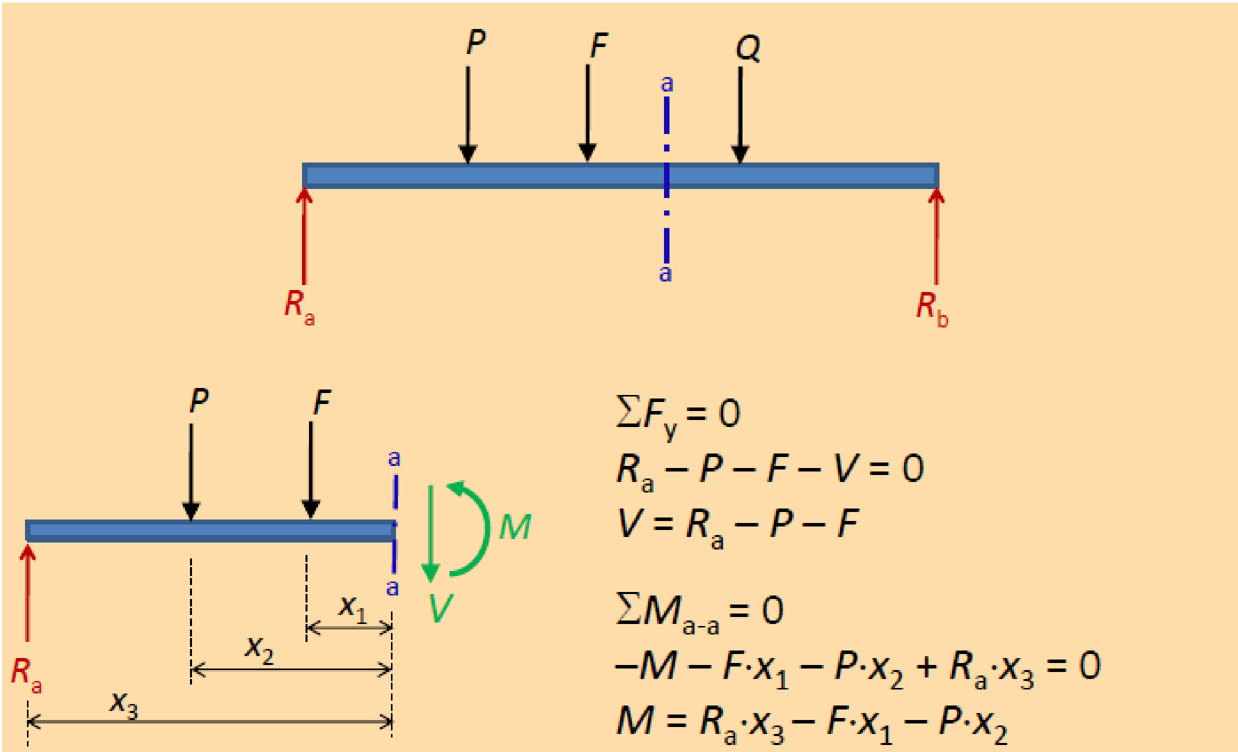
V = shear force
= the force that tends to separate the member
= balances the reaction R_A

M = bending moment
= the reaction moment at a particular point (section)
= balances the moment, $R_A.x$

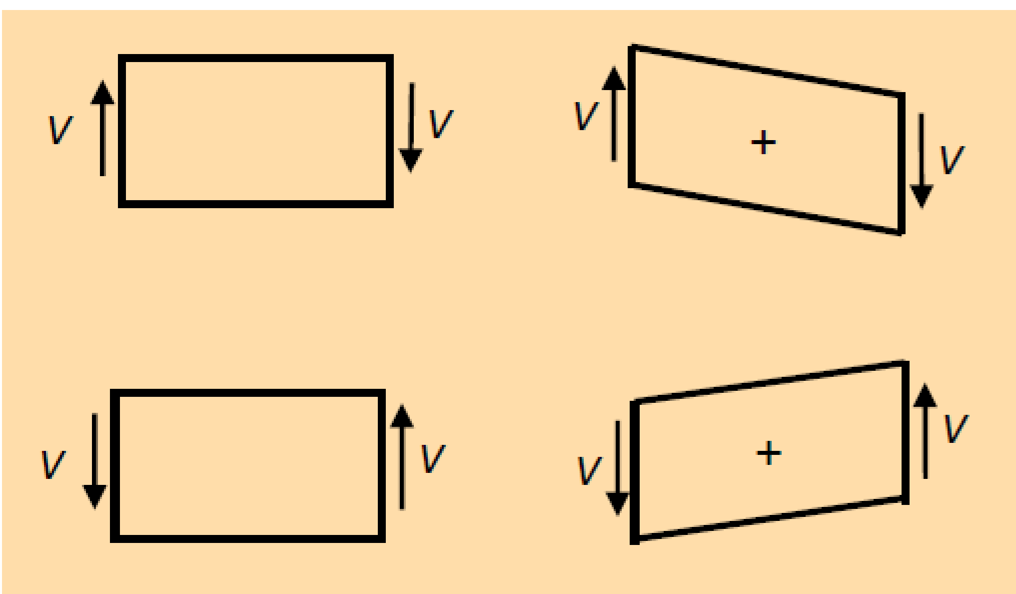
From the equilibrium equations of statics

$$+\uparrow \sum F_y = 0; \qquad R_A - V = 0 \qquad \therefore V = R_A$$

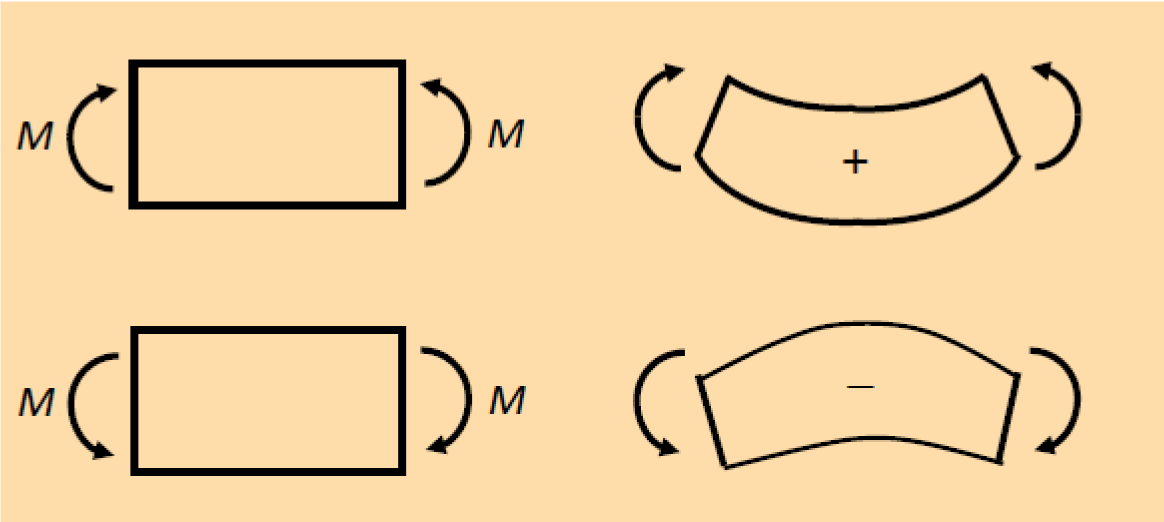
$$+\circlearrowright \sum M_{a\text{-}a} = 0; \qquad -M + R_A \cdot x = 0 \qquad \therefore M = R_A \cdot x$$



Shape deformation due to shear force:



Shape deformation due to bending moment:



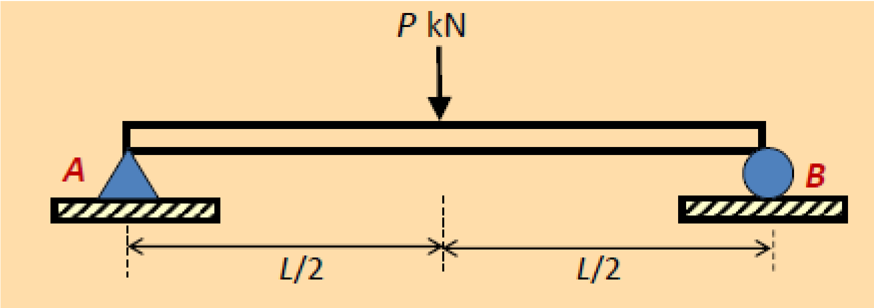
Sign Convention:

- **Positive shear force** diagram drawn **ABOVE** the beam
- **Positive bending moment** diagram drawn **BELOW** the beam

Example:05

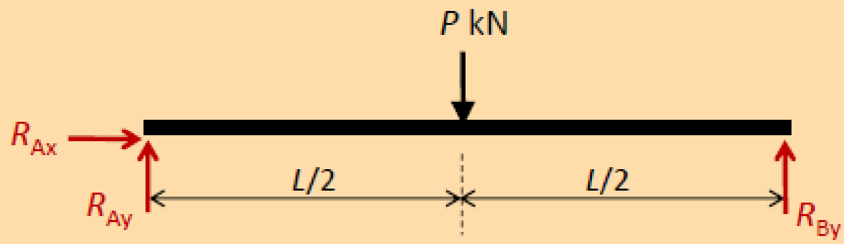
a)Calculate the shear force and bending moment for the beam subjected to a concentrated load as shown in the figure. Then, draw the shear force diagram (SFD) and bending moment diagram (BMD).

b)If P = 20 kN and L = 6 m, draw the SFD and BMD for the beam.



SOLUTION

a)



By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times L + P \times L/2 = 0$

$R_{By} = P/2$ kN

$\Sigma F_y = 0$

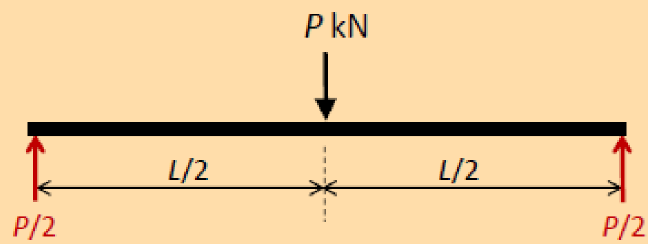
$R_{Ay} + R_{By} = P$

$R_{Ay} = P - P/2$

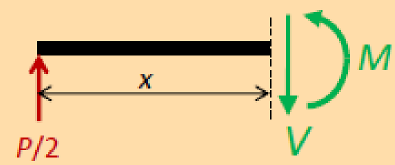
$R_{Ay} = P/2$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$



<u>Between $0 < x < L/2$:</u>



$\Sigma F_y = 0,$ $\quad -V + P/2 = 0$

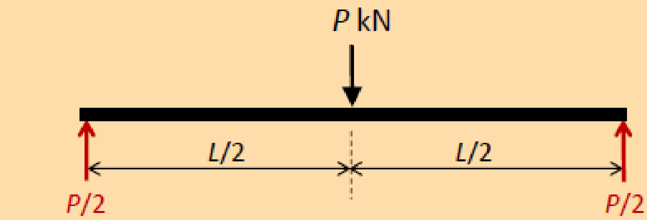
$V = P/2$ kN

$\Sigma M_{a\text{-}a} = 0,$ $\quad -M + Px/2 = 0$

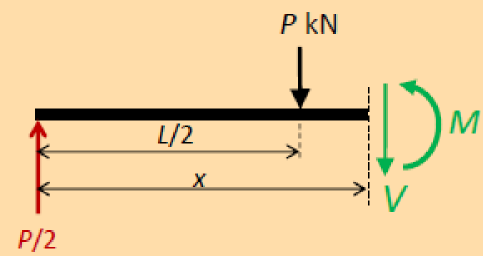
$M = Px/2$ kNm

If $x = 0$ m, $\boldsymbol{V = P/2}$ **kN** and $\boldsymbol{M = 0}$ **kNm**

If $x = L/2$ m, $\boldsymbol{V = P/2}$ **kN** and $\boldsymbol{M = PL/4}$ **kNm**



Between $L/2 < x < L$:



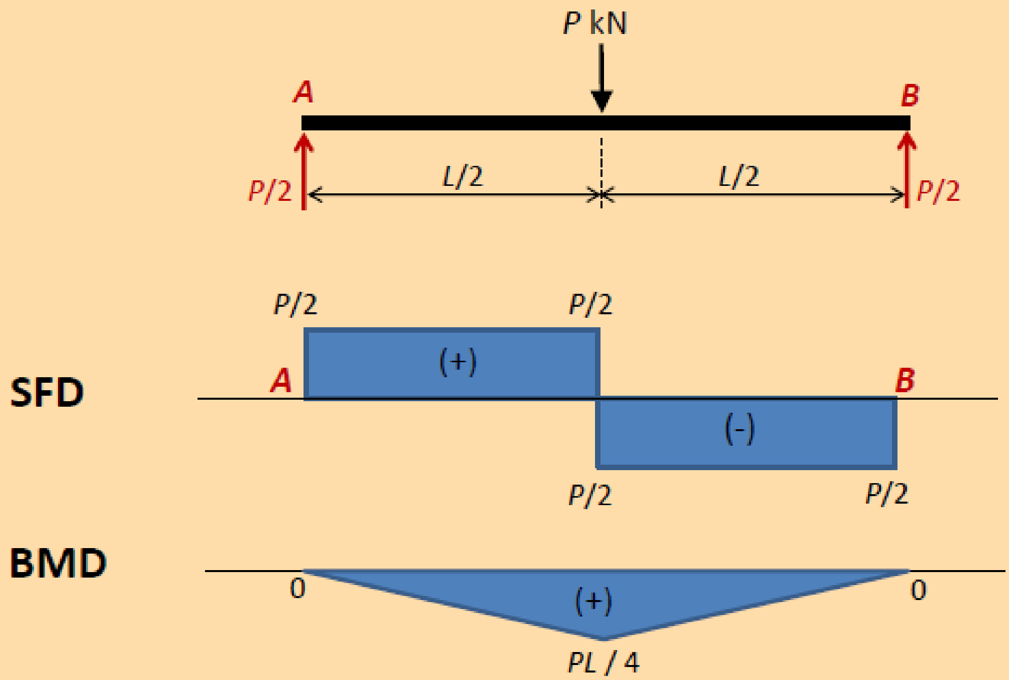
$\Sigma F_y = 0$, $\quad -V + P/2 - P = 0$

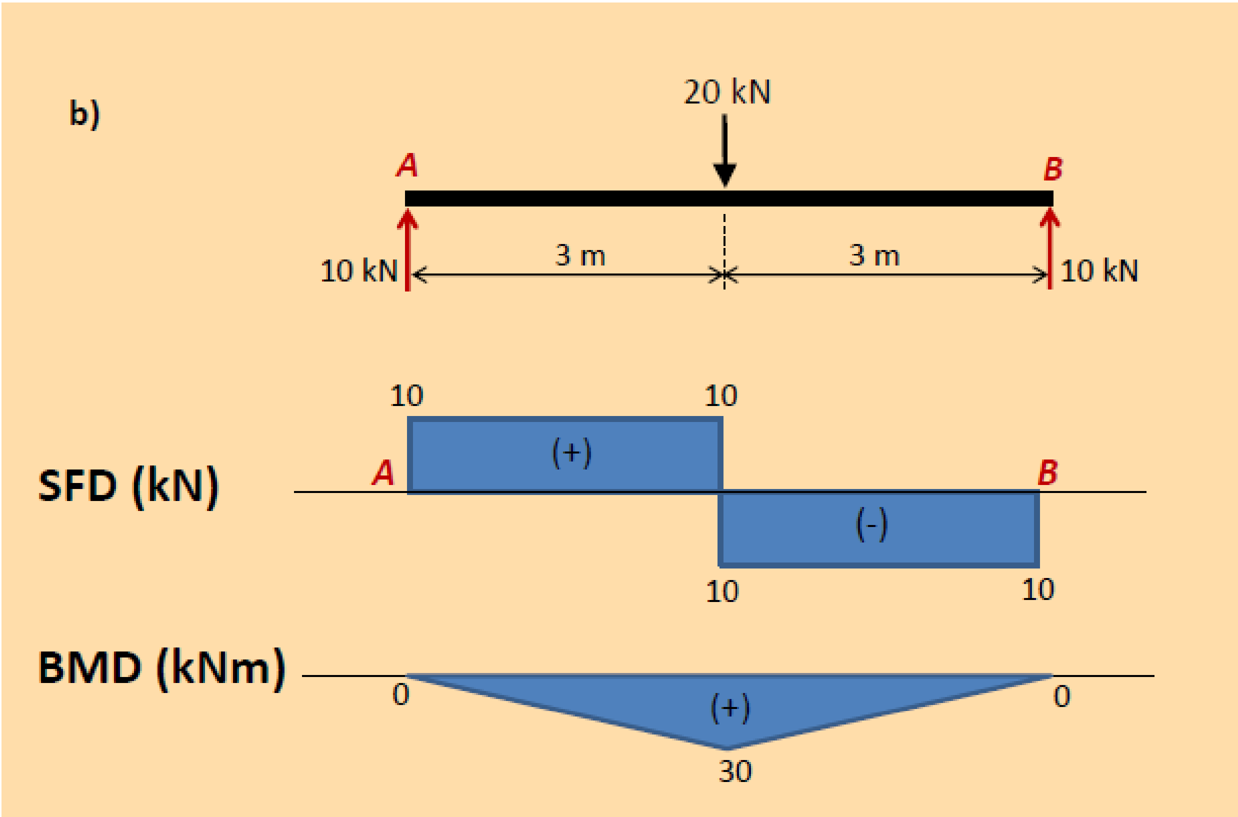
$V = -P/2$ kN

$\Sigma M_{a\text{-}a} = 0$, $\quad -M + Px/2 - P(x - L/2) = 0$

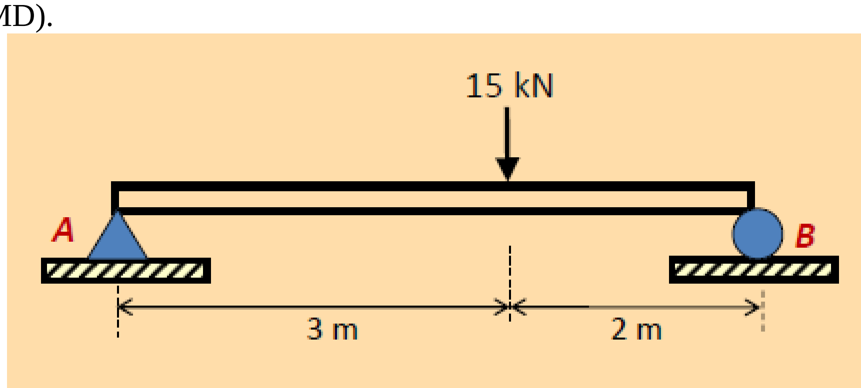
$M = PL/2 - Px/2$ kNm

If $x = L/2$ m, $\boldsymbol{V = -P/2}$ **kN** and $\boldsymbol{M = PL/4}$ **kNm**
If $x = L$ m, $\boldsymbol{V = -P/2}$ **kN** and $\boldsymbol{M = 0}$ **kNm**

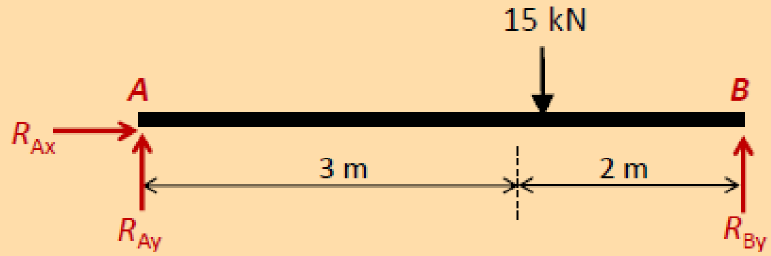




Example:06
Calculate the shear force and bending moment for the beam subjected to a concentrated load as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times 5 + 15 \times 3 = 0$

$R_{By} = 9$ kN

$\Sigma F_y = 0$

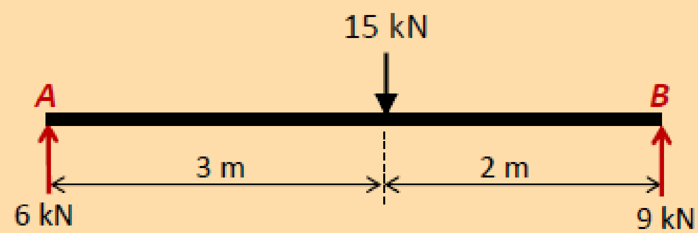
$R_{Ay} + R_{By} = 15$

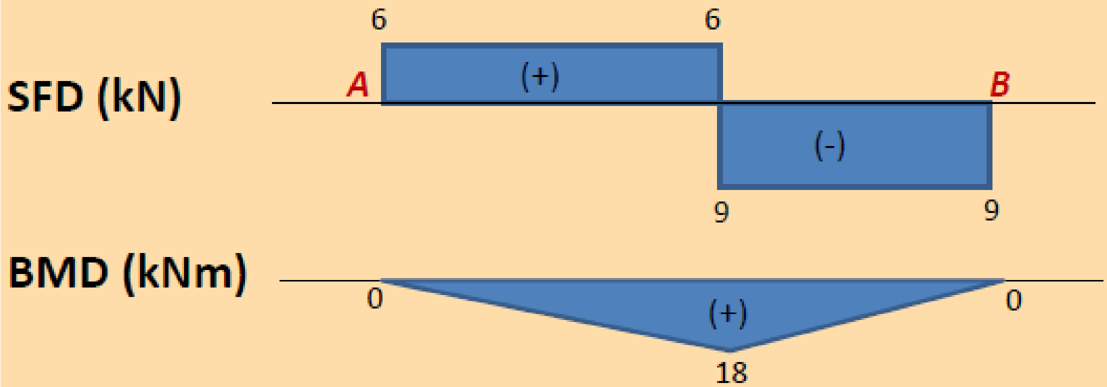
$R_{Ay} = 15 - 9$

$R_{Ay} = 6$ kN

$\Sigma F_x = 0$

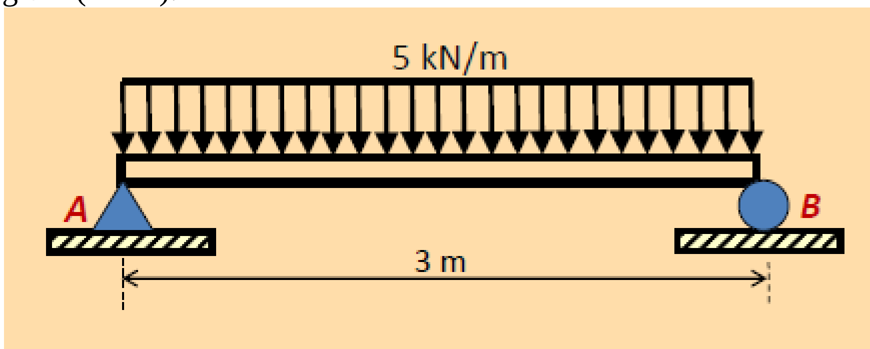
$R_{Ax} = 0$



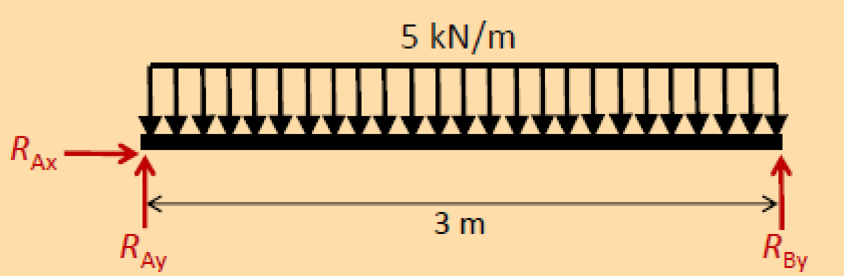


Example:07

Calculate the shear force and bending moment for the beam subjected to an uniformly distributed load as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times 3 + 5 \times 3 \times 3/2 = 0$

$R_{By} = 7.5$ kN

$\Sigma F_y = 0$

$R_{Ay} + R_{By} = 5 \times 3$

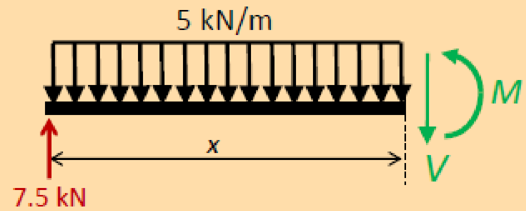
$R_{Ay} = 15 - 7.5$

$R_{Ay} = 7.5$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$

These results for V and M can be checked by noting that dV/dx = -w. This is correct, since positive w acts downward. Also, notice that dM/dx = V. The maximum moments occurs when dM/dx = V = 0.



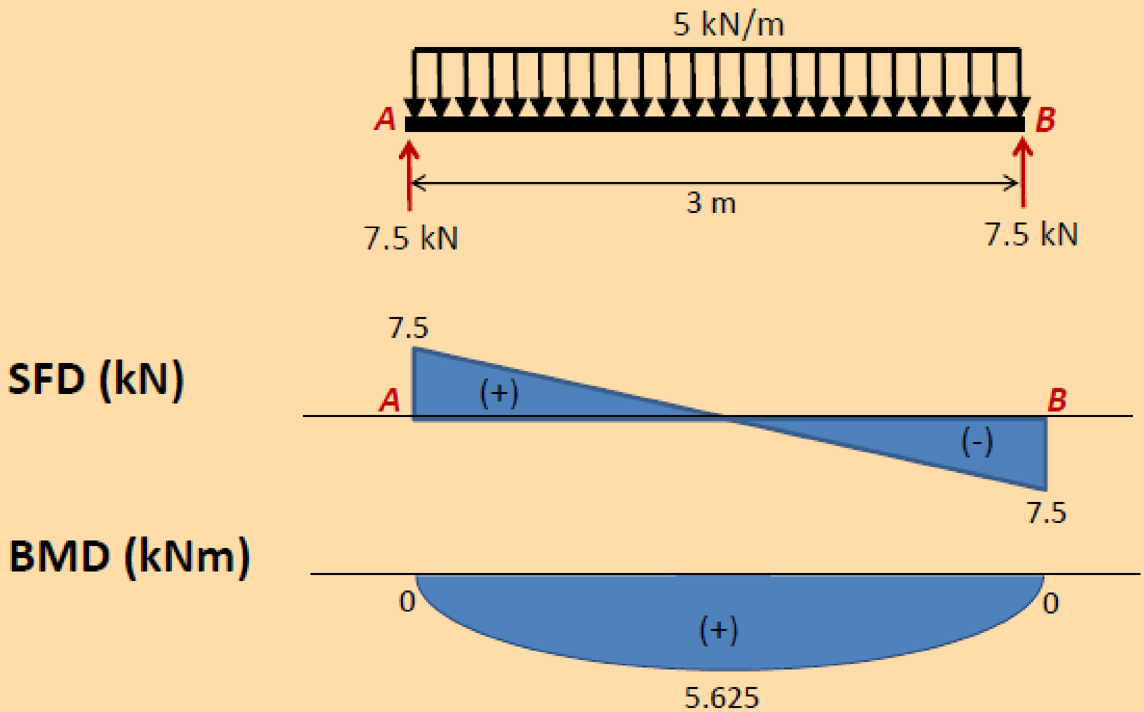
$\Sigma M_{a\text{-}a} = 0,\quad -M + 7.5x - 5x\,(x/2) = 0$

$M = 7.5x - 5x^2/2$

M = maximum when $\dfrac{dM}{dx} = 0$

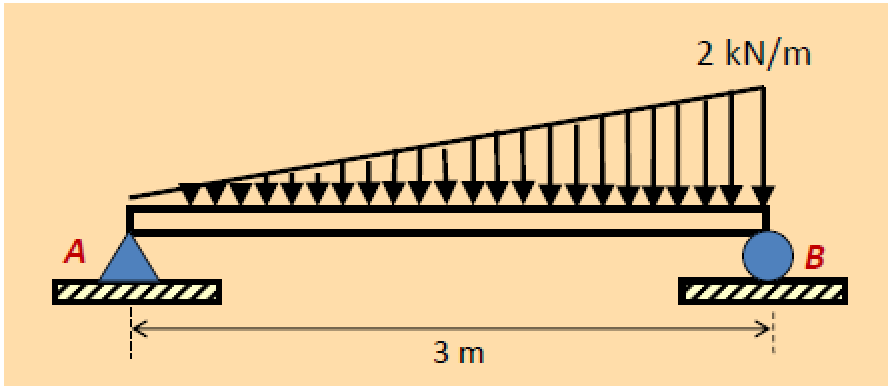
$\dfrac{dM}{dx} = 7.5 - 5x = 0 \qquad \therefore\ x = 1.5$ m

Therefore, M_{max} = 5.625 kNm

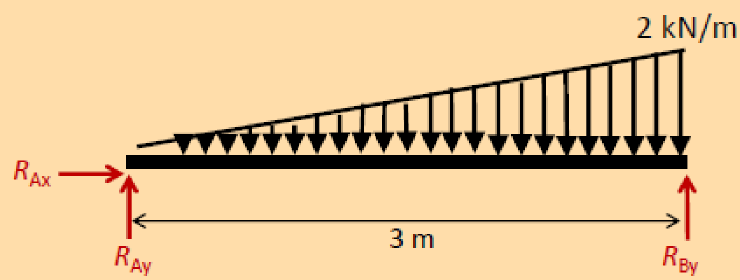


Example:08

Calculate the shear force and bending moment for the beam subjected to the loads as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at A:

$\Sigma M_A = 0$

$2 \times 3/2 \times 3 \times 2/3 - R_{By} \times 3 = 0$

$R_{By} = 2$ kN

$\Sigma F_y = 0$

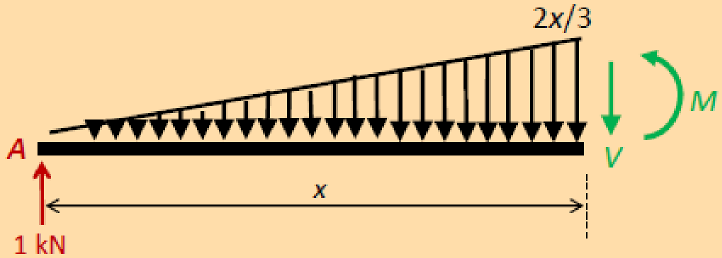
$R_{Ay} + R_{By} = 2 \times 3/2$

$R_{Ay} = 3 - 2$

$R_{Ay} = 1$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$



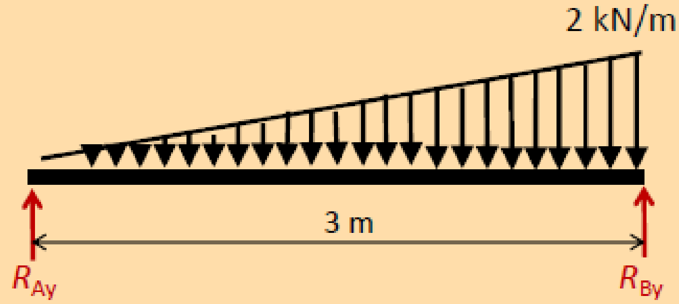
$$1 - 2x/3(x)(1/2) - V = 0$$
$$V = 1 - 2x^2/6$$
If $x = 0$, $V = 1$ kN and $x = 3$, $V = -2$ kN

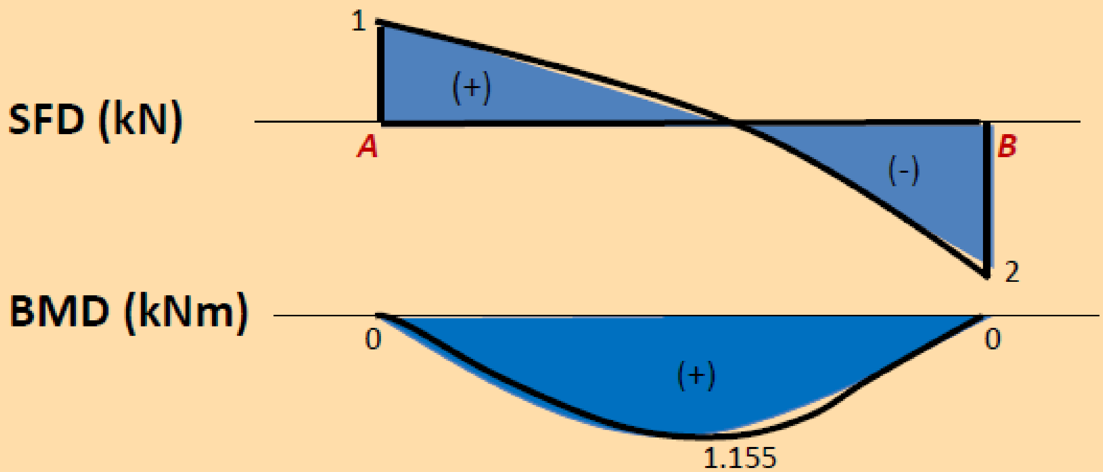
$$-M + 1 \times x - 2x/3(x)(1/2)\,(x/3) = 0$$
$$M = x - x^3/9$$

M = maximum when $\dfrac{dM}{dx} = 0$

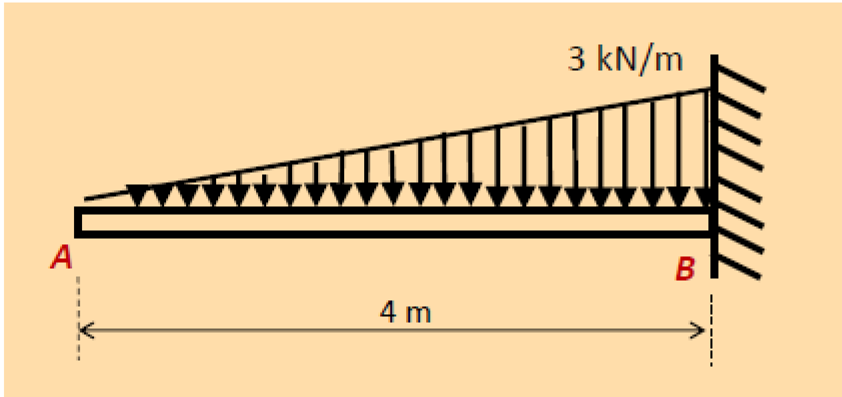
$$\frac{dM}{dx} = 1 - \frac{3x^2}{9} = 0$$
$$x^2 = \frac{9}{3}$$
$$x = \frac{3}{\sqrt{3}} = 1.732m$$

Therefore, M_{max} = 1.155 kNm

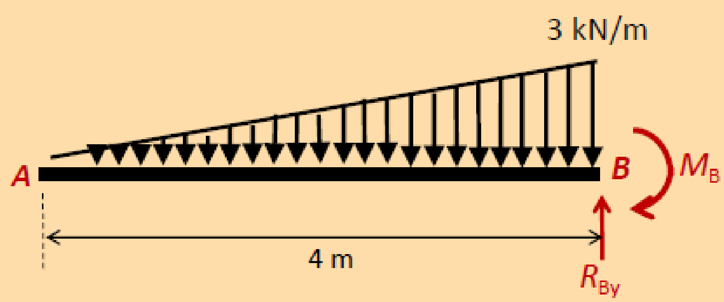




Example:09
Calculate the shear force and bending moment for the beam subjected to the loads as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at B:

$\Sigma M_B = 0$

$M_B = 3 \times 4/2 \times 4/3$

$M_B = 8$ kNm

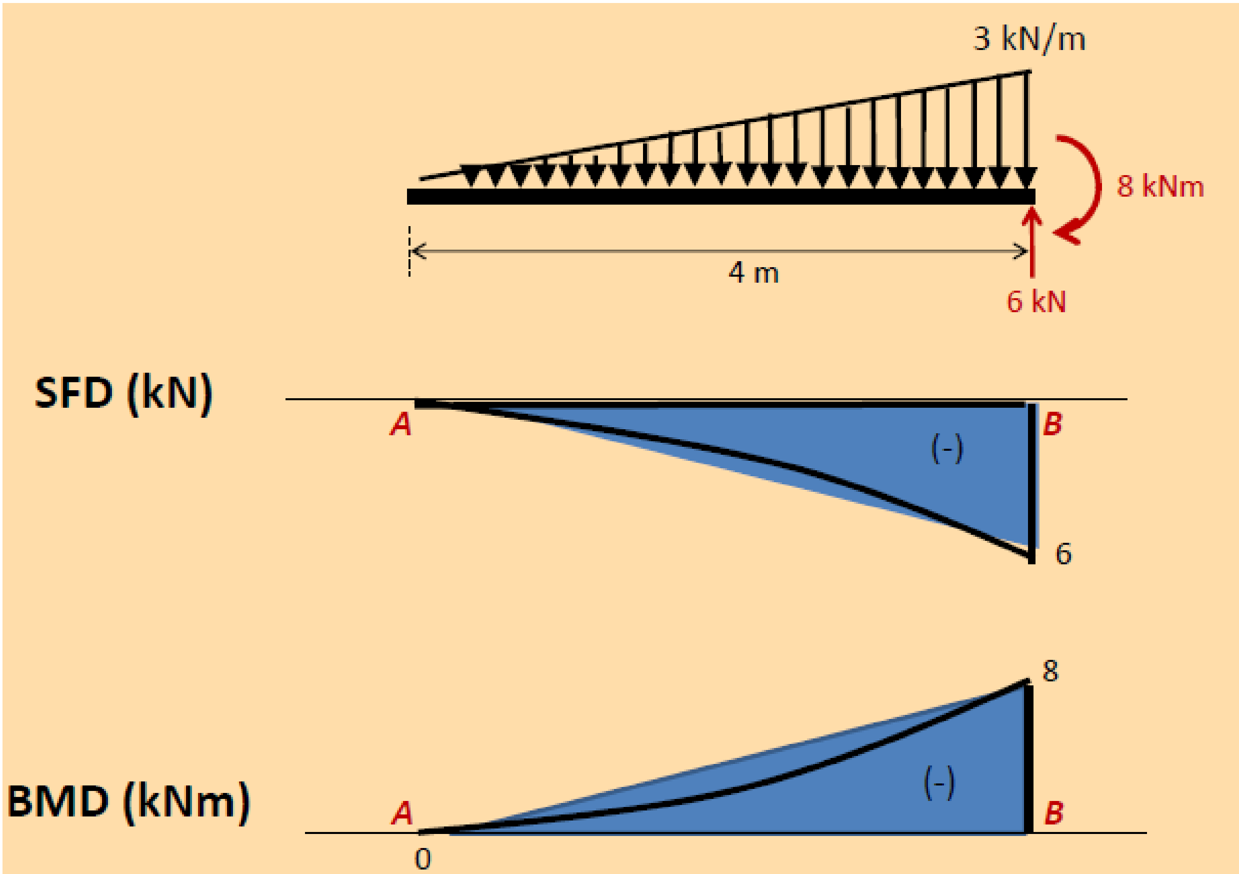
$\Sigma F_y = 0$

$R_{By} = 3 \times 4/2$

$R_{By} = 6$ kN

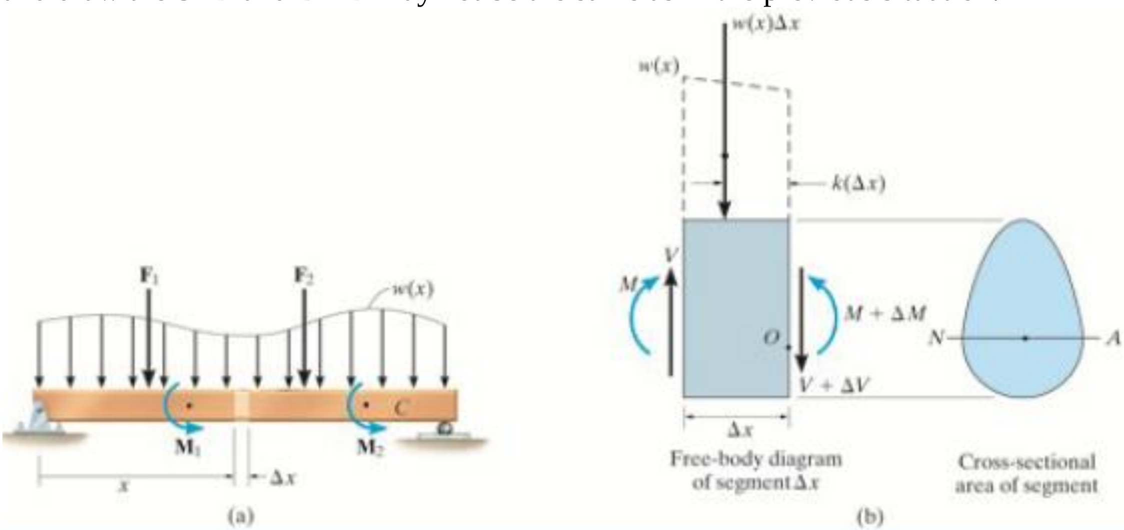
$\Sigma F_x = 0$

$R_{Bx} = 0$



RELATIONSHIP BETWEEN LOAD, SHEAR FORCE & BENDING MOMENT

When a beam is subjected to two or more concentrated or distributed load, the way to calculate and draw the SFD and BMD may not be the same as in the previous situation.



REGION OF DISTRIBUTED LOAD

$\Sigma F_y = 0; \quad V - w(x)\Delta x - (V + \Delta V) = 0$

$\Delta V = w(x)\Delta x$

$\Sigma M_0 = 0;$

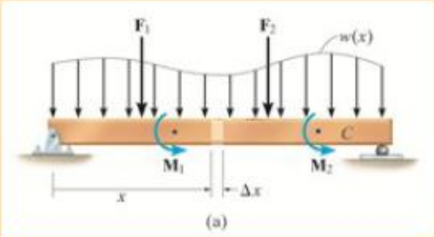
$-V\,\Delta x - M + w(x)\Delta x[k\,\Delta x] + (M + \Delta M) = 0$

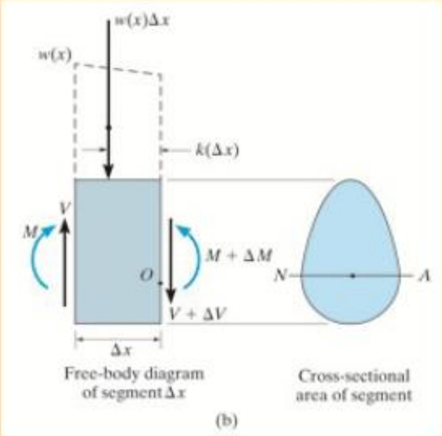
$\Delta M = V\,\Delta x - w(x)k\Delta x^2$

Dividing by Δx and taking the limit as $\Delta x = 0$, the above two equations become:

Slope of the shear diagram at each point $\dfrac{dV}{dx} = -w(x)$ distributed load intensity at each point

Slope of moment diagram at each point $\dfrac{dM}{dx} = V$ Shear at each point

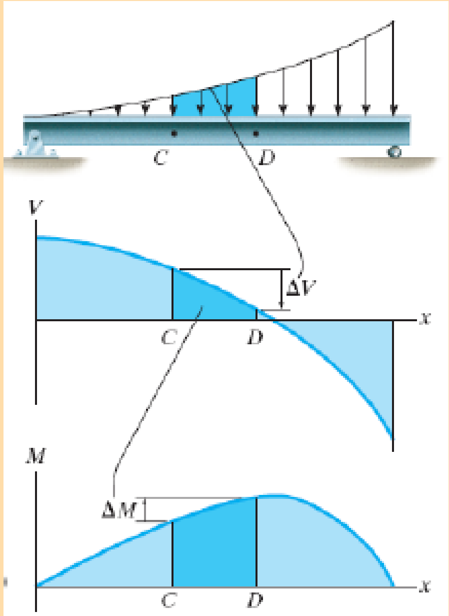




- We can integrate these areas between any two points to get change in shear and moment.

Change in shear $\Delta V = -\int w(x)dx$ Area under distributed loading

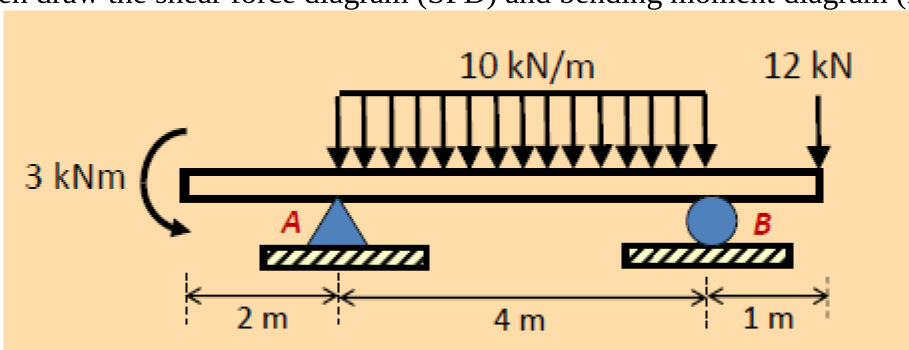
Change in moment $\Delta M = \int V(x)dx$ Area under shear diagram



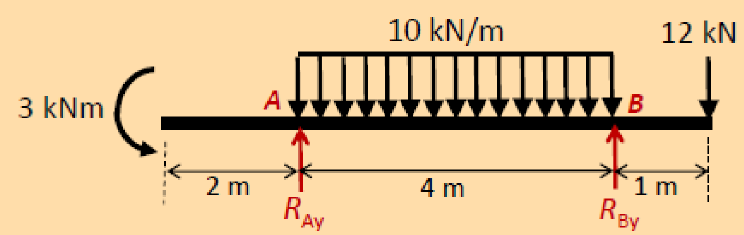
HINTS TO KNOW

- Slope of bending moment always determined by the shape of shear force lines. The changes in slope (sagging or hogging also depends on the changes in shear force values)
- When shear force intersects BMD axis, there is a maximum moment
- When SF maximum, BM minimum and vice versa
- SFD and BMD always start and end with zero values (unless at the point where there is a moment/couple)
- When a moment/couple acting: –Clockwise (↓) (+), Anticlockwise (↑) (-)

Example:10
Calculate the shear force and bending moment for the beam subjected to the loads as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times 4 - 3 + 10 \times 4 \times 4/2 + 12 \times 5 = 0$

$R_{By} = 34.25$ kN

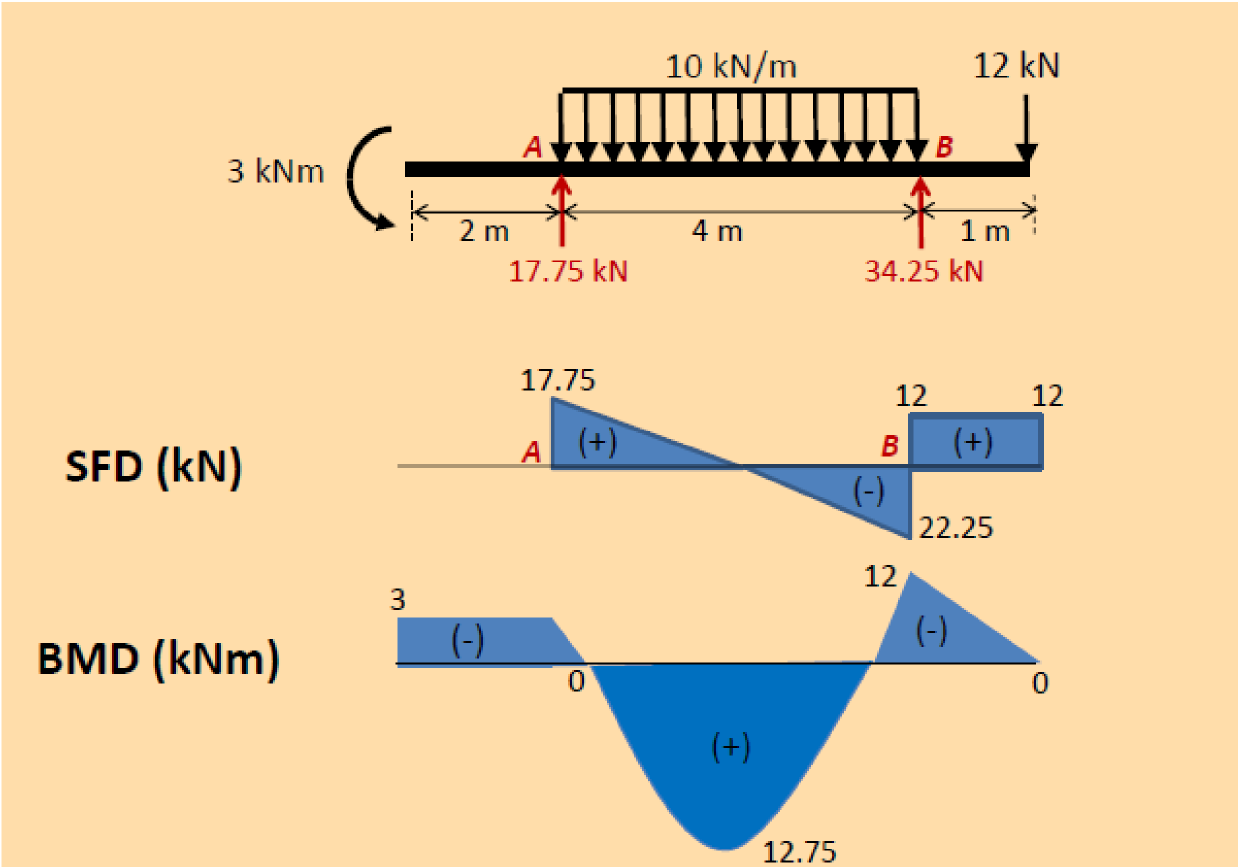
$\Sigma F_y = 0$

$R_{Ay} + R_{By} = 10 \times 4 + 12$

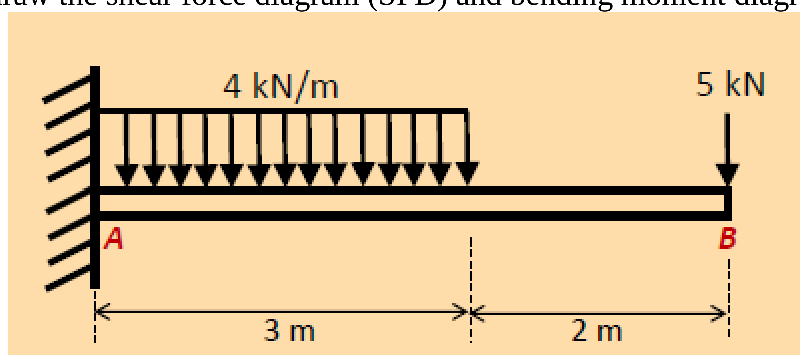
$R_{Ay} = 17.75$ kN

$\Sigma F_x = 0$

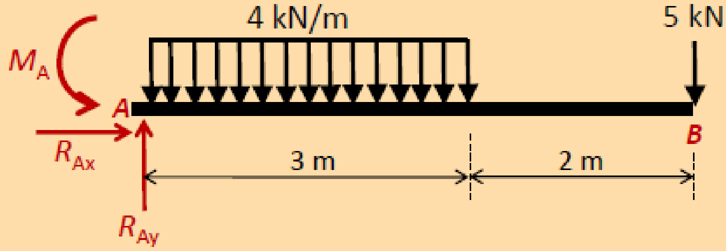
$R_{Ax} = 0$



Example:11
Calculate the shear force and bending moment for the beam subjected to the loads as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at A:

$\Sigma M_A = 0$

$-M_A + 4 \times 3 \times 3/2 + 5 \times 5 = 0$

$M_A = 43$ kNm

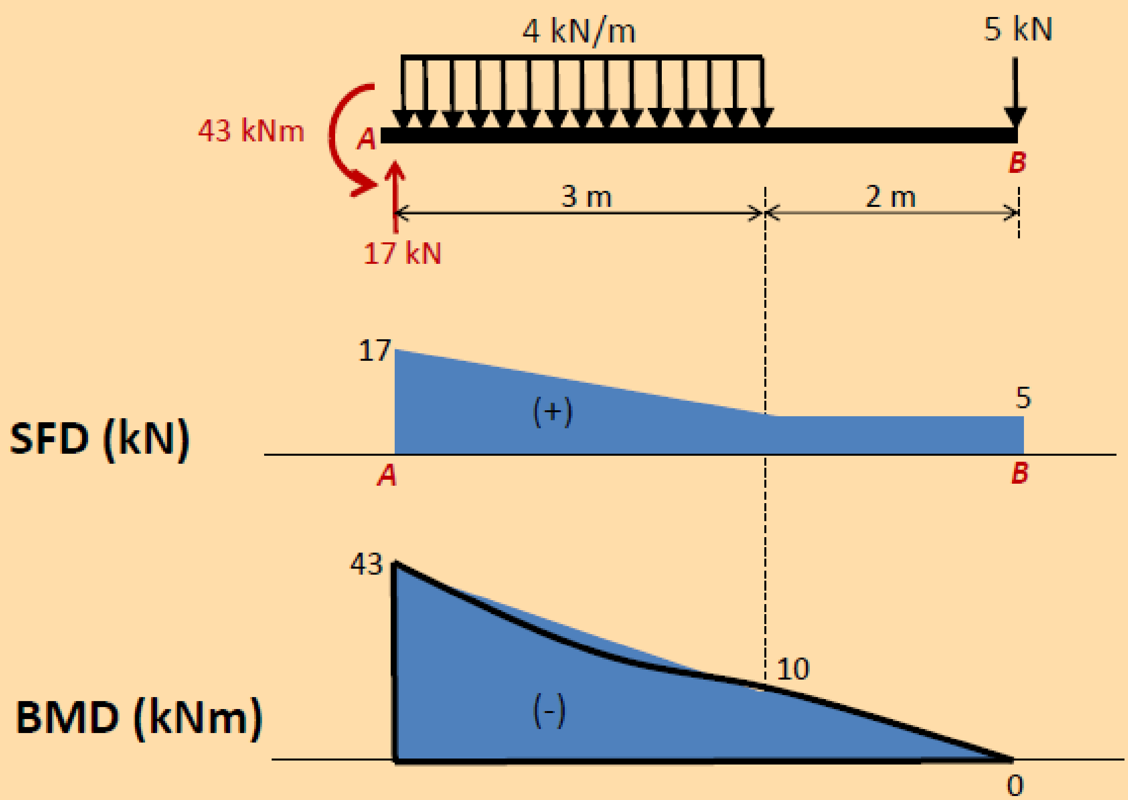
$\Sigma F_y = 0$

$R_{Ay} = 4 \times 3 + 5$

$R_{Ay} = 17$ kN

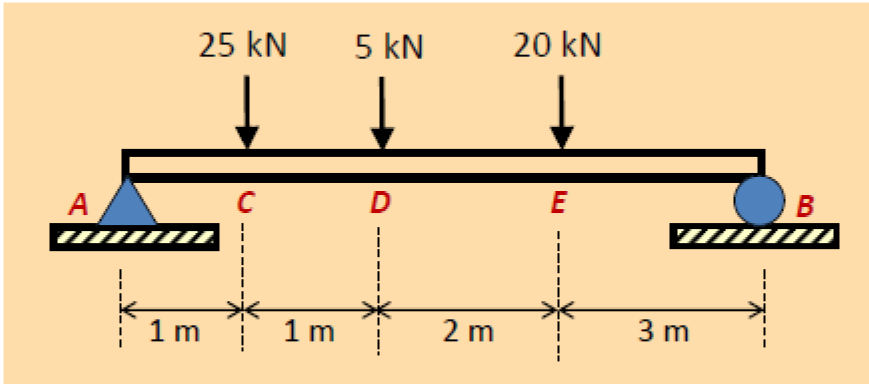
$\Sigma F_x = 0$

$R_{Ax} = 0$

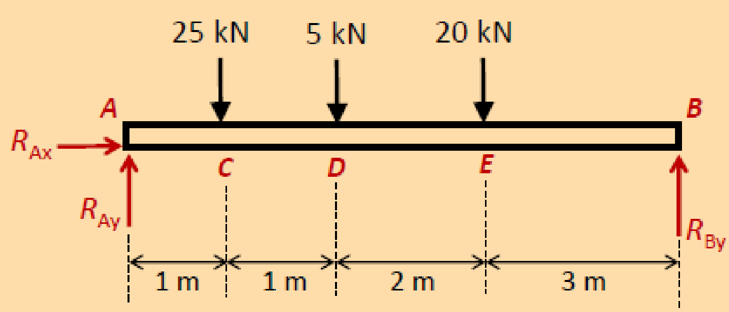


Example:12

Calculate the shear force and bending moment for the beam subjected to the loads as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at A:

$\Sigma M_A = 0$

$25 \times 1 + 5 \times 2 + 20 \times 4 - R_{By} \times 7 = 0$

$R_{By} = 16.43$ kN

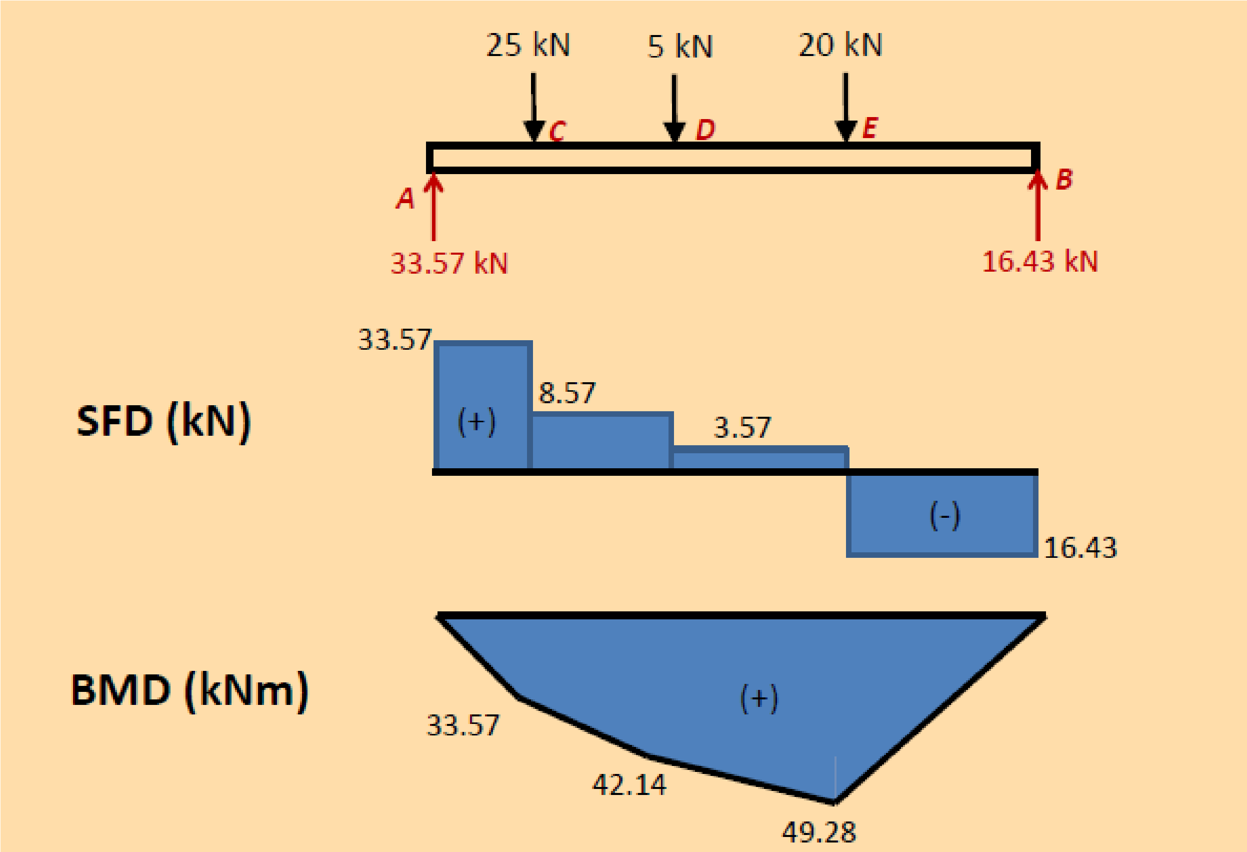
$\Sigma F_y = 0$

$R_{Ay} + R_{By} = 25 + 5 + 20$

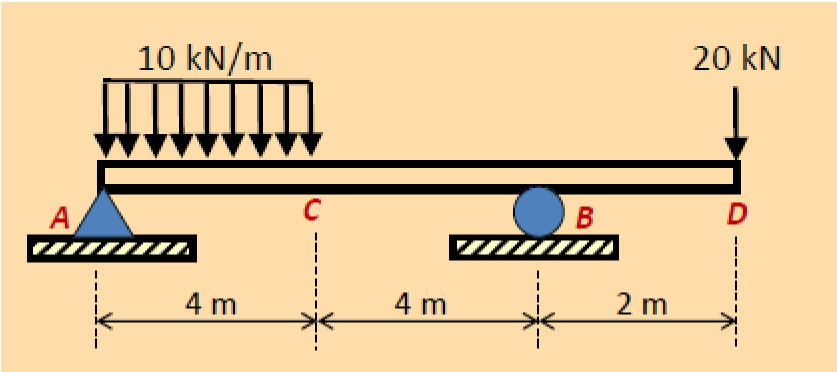
$R_{Ay} = 33.57$ kN

$\Sigma F_x = 0$

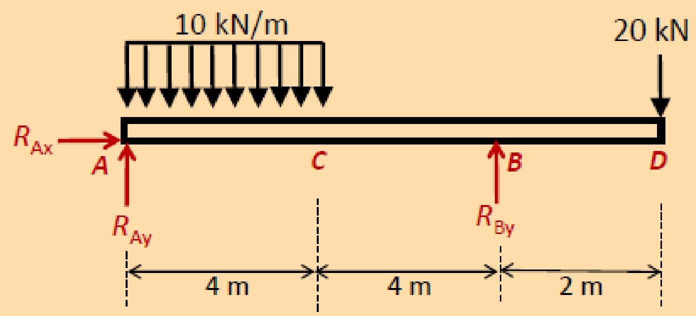
$R_{Ax} = 0$



Example:13
Calculate the shear force and bending moment for the beam subjected to the loads as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at A:

$\Sigma M_A = 0$

$10 \times 4 \times 2 + 20 \times 10 - R_{By} \times 8 = 0$

$R_{By} = 35$ kN

$\Sigma F_y = 0$

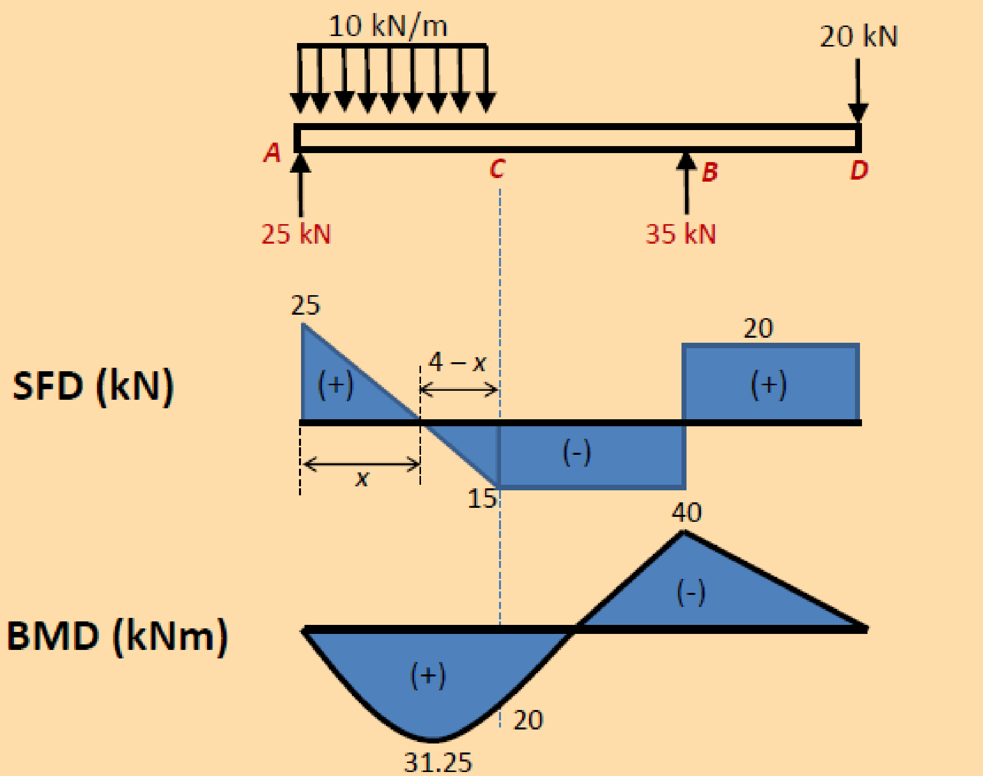
$R_{Ay} + R_{By} = 10 \times 4 + 20$

$R_{Ay} = 60 - 35$

$R_{Ay} = 25$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$



$$\frac{x}{25} = \frac{4-x}{15}$$

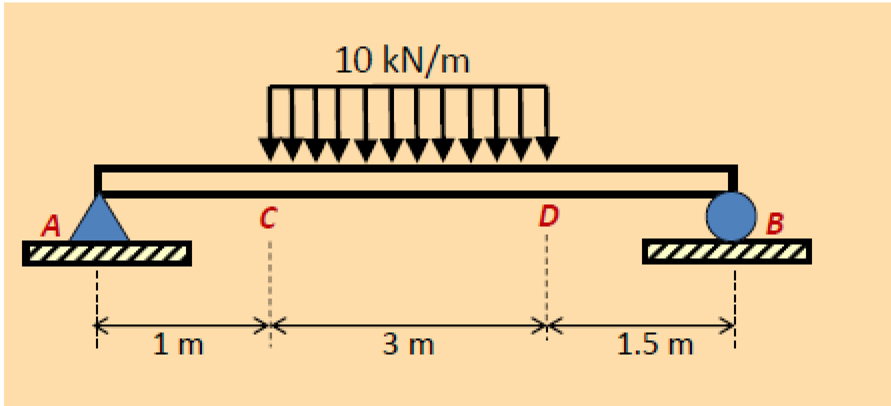
$$15x = 100 - 25x$$

$$40x = 100$$

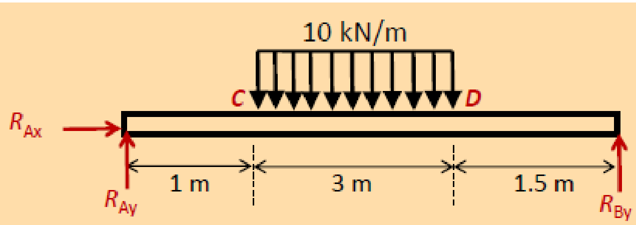
$$x = 2.5$$

Example:14

Calculate the shear force and bending moment for the beam subjected to the loads as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at A:

$\Sigma M_A = 0$

$10 \times 3 \times 2.5 - R_{By} \times 5.5 = 0$

$R_{By} = 13.64$ kN

$\Sigma F_y = 0$

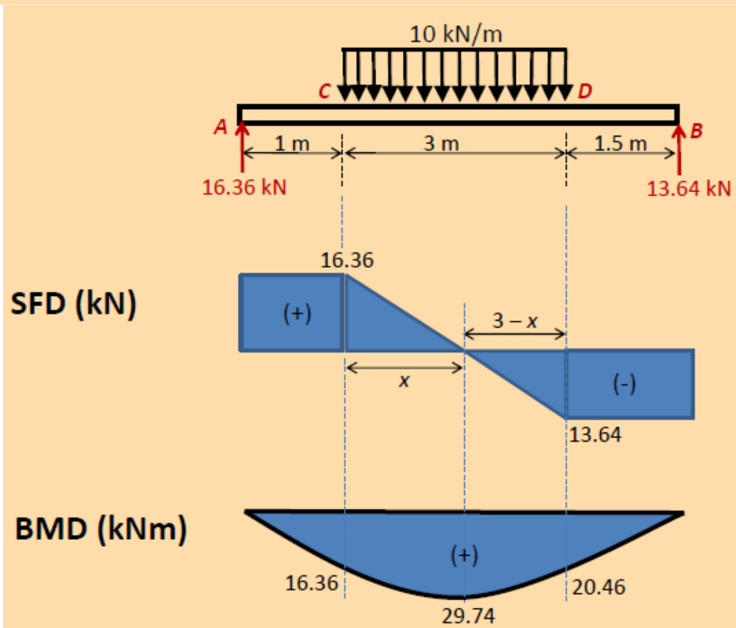
$R_{Ay} + R_{By} = 10 \times 3$

$R_{Ay} = 30 - 13.64$

$R_{Ay} = 16.36$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$



$$\frac{x}{16.36} = \frac{3-x}{13.64}$$

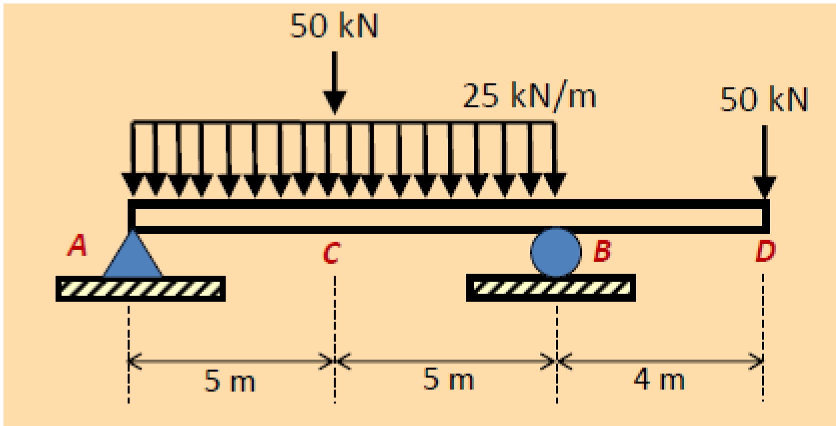
$$13.64x = 49.08 - 16.36x$$

$$30x = 49.08$$

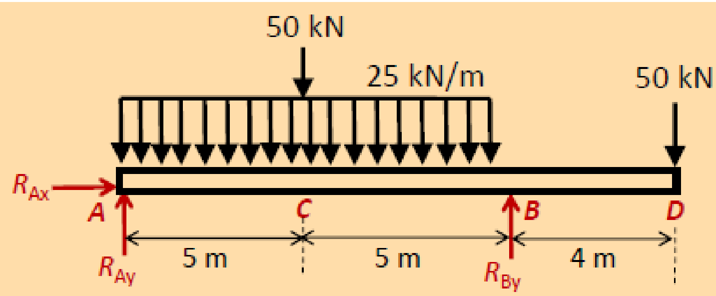
$$x = 1.636$$

Example:15

Calculate the shear force and bending moment for the beam subjected to the loads as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



By taking the moment at A:

$\Sigma M_A = 0$

$25 \times 10 \times 5 + 50 \times 5 + 50 \times 14 - R_{By} \times 10 = 0$

$R_{By} = 220$ kN

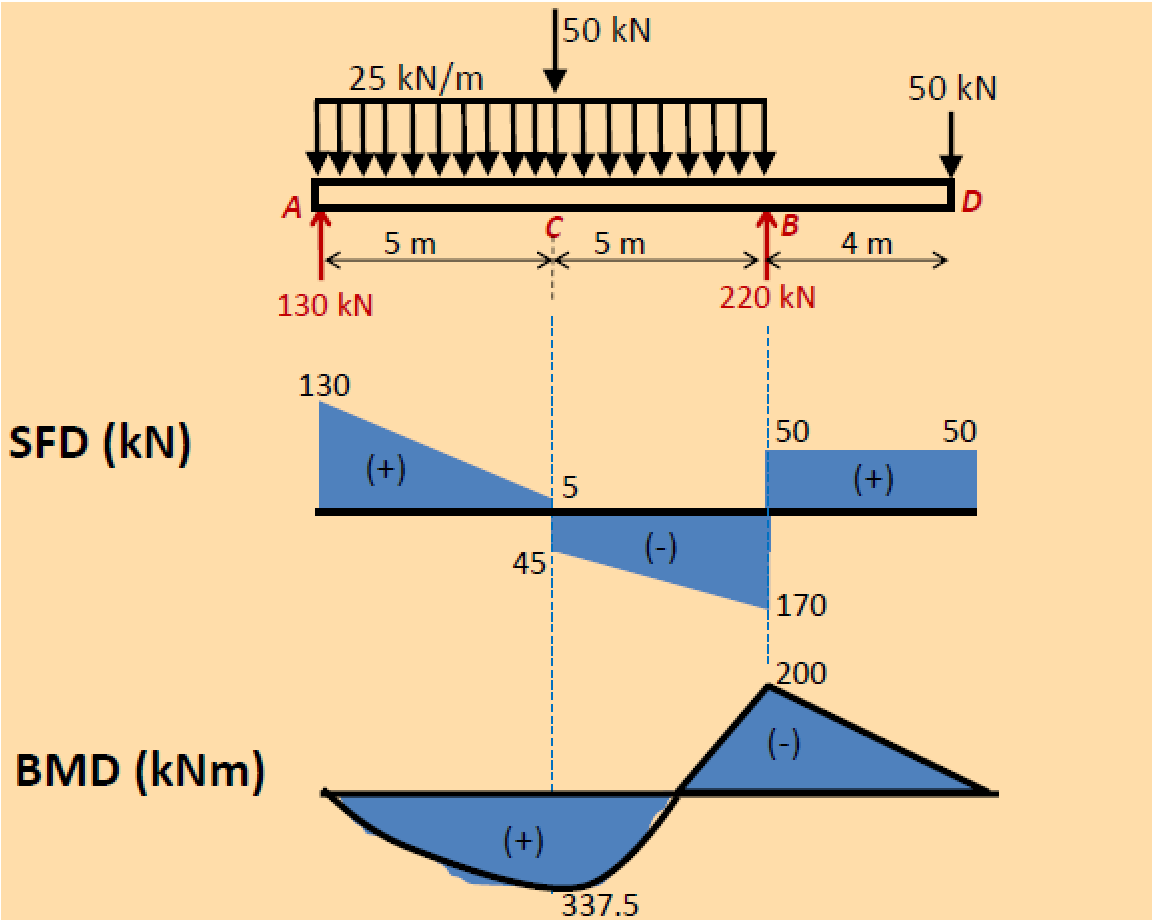
$\Sigma F_y = 0$

$R_{Ay} + R_{By} = 25 \times 10 + 50 + 50$

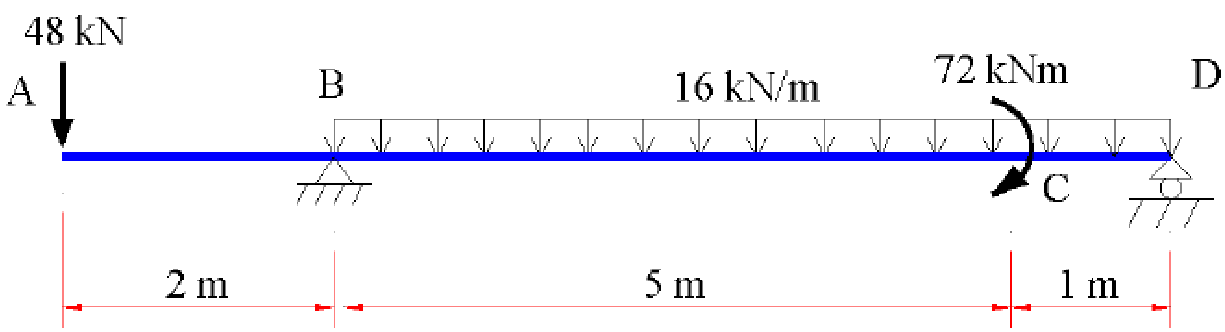
$R_{Ay} = 130$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$

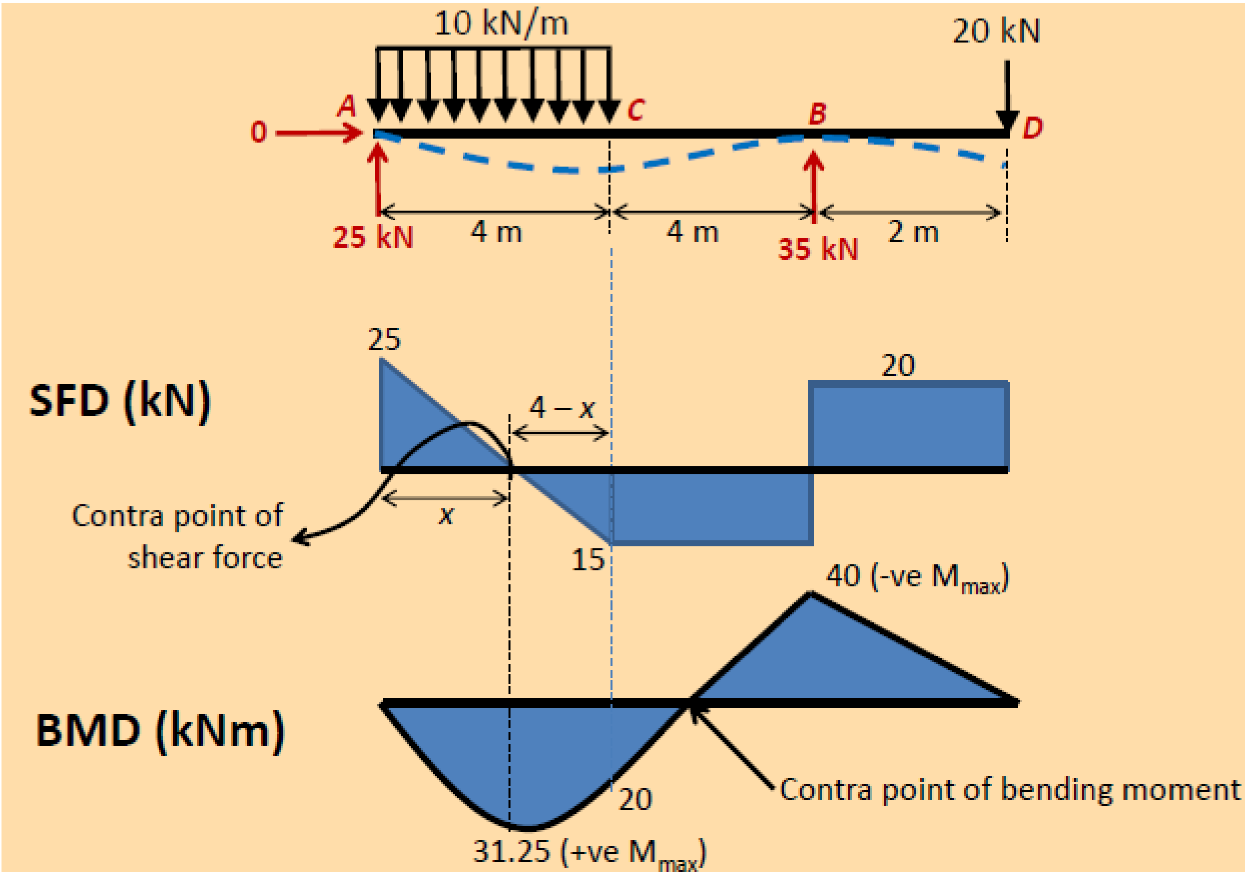


Example:16(discussion)
Draw s.f and b.m diagram,given data in a figure below.



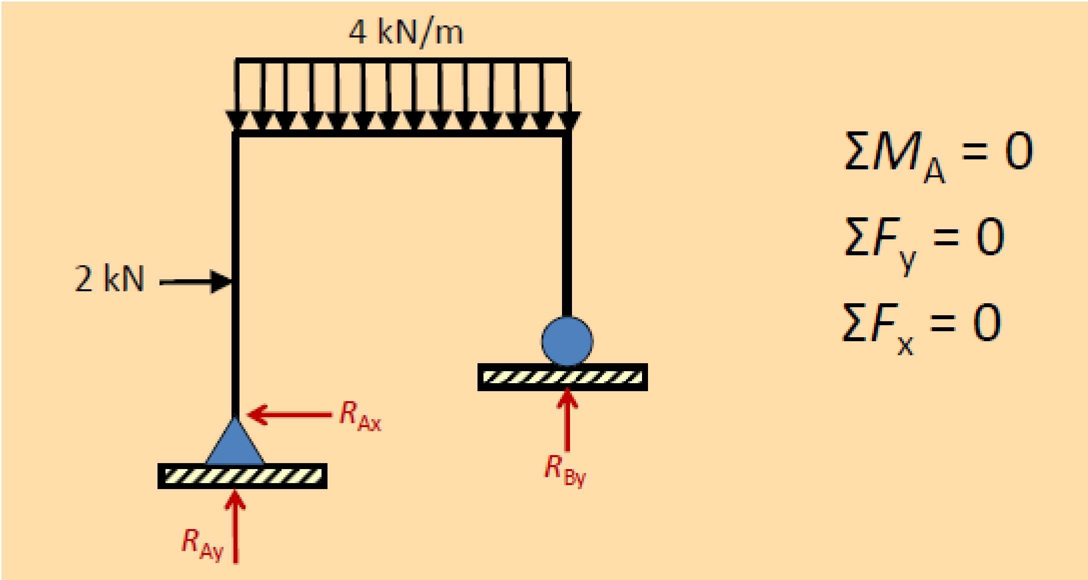
CONTRA POINT OF SHEAR FORCE & BENDING MOMENT
(Point of Contraflexure)

- Contra point is a place where positive shear force/bending moment shifting to the negative region or vice-versa.
- Contra point for shear: ***V = 0***
- Contra point for moment: ***M = 0***
- When shear force is ***zero***, the moment is ***maximum***.
- ***Maximum shear force*** usually occur at the ***support / concentrated load***.

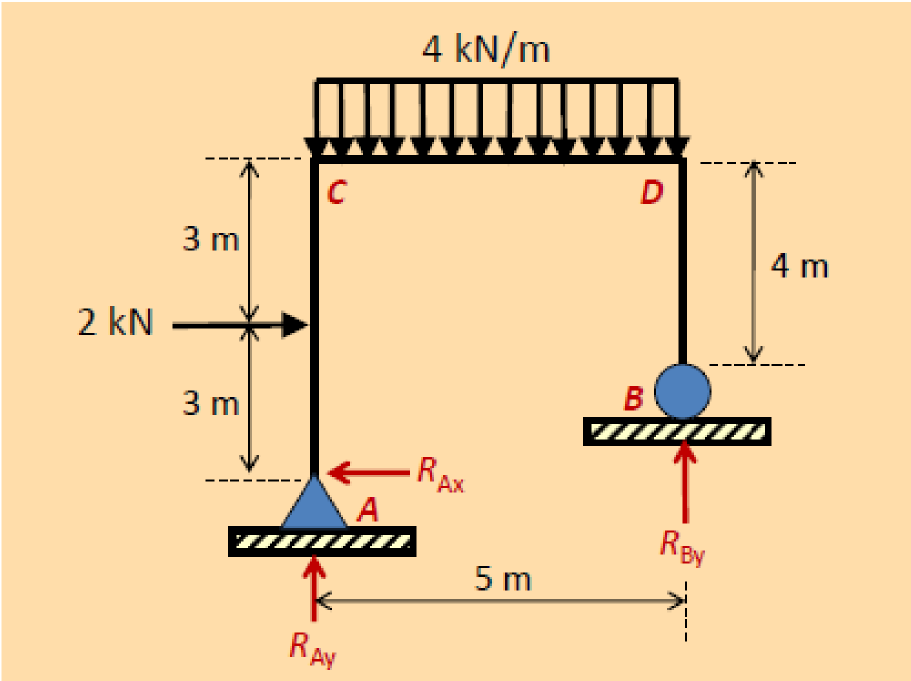


STATICALLY DETERMINATE FRAMES

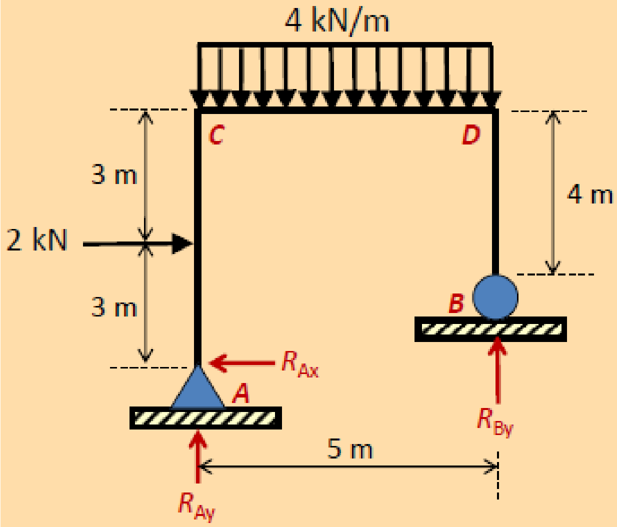
For a frame to be statically determinate, the number of unknown (reactions) must be able to solved using the equations of equilibrium.



Example:17
Calculate the shear force and bending moment for the frame subjected to the loads as shown in the figure, then draw the shear force diagram (SFD) and bending moment diagram (BMD).



SOLUTION



$\Sigma M_A = 0$

$4 \times 5 \times 2.5 + 2 \times 3 - R_{By} \times 5 = 0$

$R_{By} = 11.2$ kN

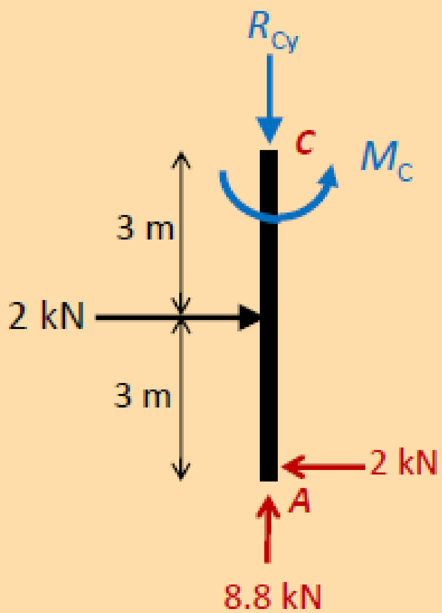
$\Sigma F_y = 0$

$R_{Ay} + R_{By} = 4 \times 5$

$R_{Ay} = 8.8$ kN

$\Sigma F_x = 0$

$R_{Ax} = 2$ kN

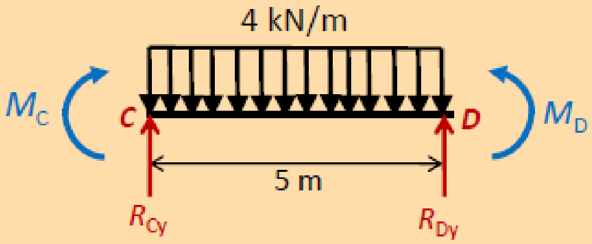


$\Sigma M_A = 0$: $2 \times 3 - M_C = 0$

$\therefore\ M_C = 6$ kNm

$\Sigma F_y = 0$

$\therefore\ R_{cy} = 8.8$ kN



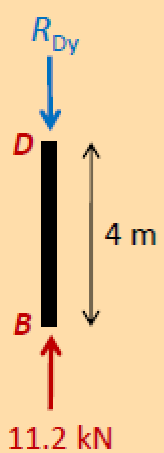
$\Sigma F_y = 0$: $R_{Cy} + R_{Dy} = 4 \times 5$

$R_{Dy} = 20 - 8.8 = 11.2$ kN

$\Sigma M_C = 0$:

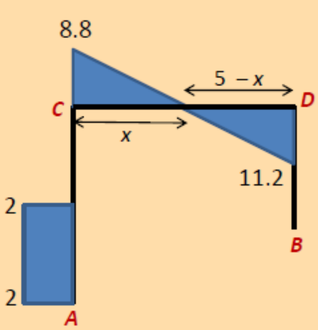
$M_C + 4 \times 5 \times 2.5 - R_{Dy} \times 5 - M_D = 0$

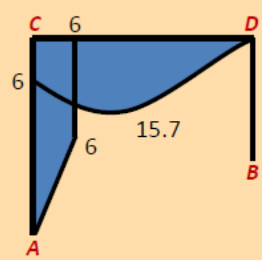
$M_D = 0$ kNm



$\Sigma F_y = 0$:

$R_{Dy} = 11.2$ kN





$$\frac{x}{8.8} = \frac{5-x}{11.2}$$

$$11.2x = 44 - 8.8x$$

$$20x = 44$$

$$x = 2.2$$

$M_{max} = 8.8 \times 2.2 \times 0.5 + 6 = 15.7$ kNm

SFD (kN)

BMD (kNm)

SUPPLEMENTARY QUESTIONS

Question 1: A cantilever beam is subject to a uniformly distributed load of *w* N per meter length. Write the equation of the shearing force and bending moment at any point *x* along its length *l.* Also sketch the shearing force and bending moment diagrams.

Question 2: A simply supported beam which is 4m long is subjected to a uniformly distributed vertical load of 1600 N/m. Draw the ***shearing force [S.F]***and ***bending moment [B.M]***diagrams.

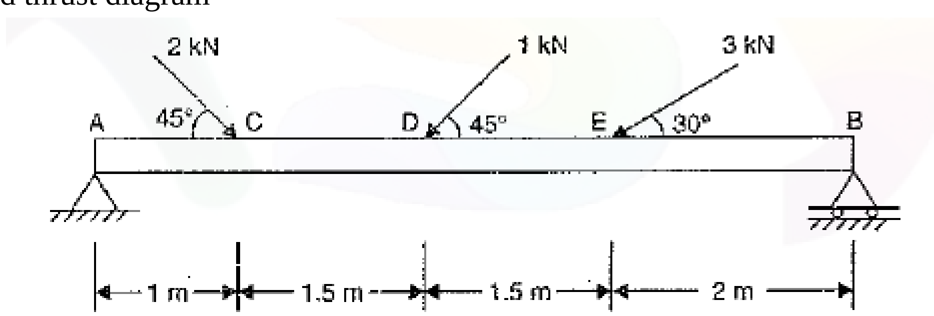
Question 3:A simply supported beam carries a vertical load that is increasing from zero at one end to a maximum of 8000 N/m at the other end. Draw the shearing force and bending moment diagrams.

Question 4: A beam which is 4 m long is simply supported over a span of 2m and overhangs both supports by a span of 1 m on either side. The right-hand overhanging portion carries a uniformly distributed load of 80 kN/m and a concentrated load of 20 kN at the extreme end while the left-hand overhanging portion carries a uniformly distributed load of 40 kN/m and a concentrated load of 30 kN at the extreme end in addition to a concentrated load of 160 kN at its mid-span. Draw to scale the S.F and B.M diagrams and find how much may be added to the load at mid-span without increasing the maximum B.M on the beam.

Question 5: A horizontal beam AB is 8 m long and carries a total uniformly distributed load of 300 kN. The beam is supported at the end A and at a point C which is at a distant *x* from the other end B. Determine the value of *x* if the mid-point of the beam is to be a point of inflexion and for this arrangement draw the S.F and B.M diagrams, indicating the principal numerical values of each.

Question 6: A girder 10 m long, carrying a uniformly distributed of *w* N/m, is to be supported on two piers 6 m apart so that the greatest bending moment on the girder shall be as small as possible. Find the distances of the piers from the ends and the maximum bending moment

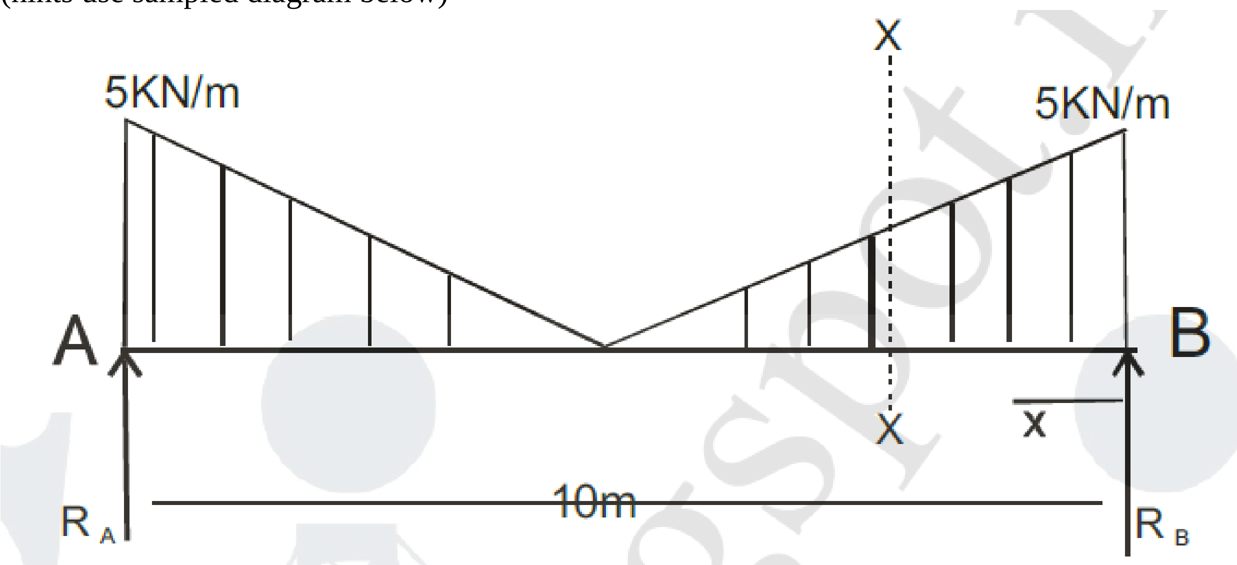
Question 7 A beam is loaded as shown in figure. Find the reactions at A and B also draw S.F and B.M and thrust diagram



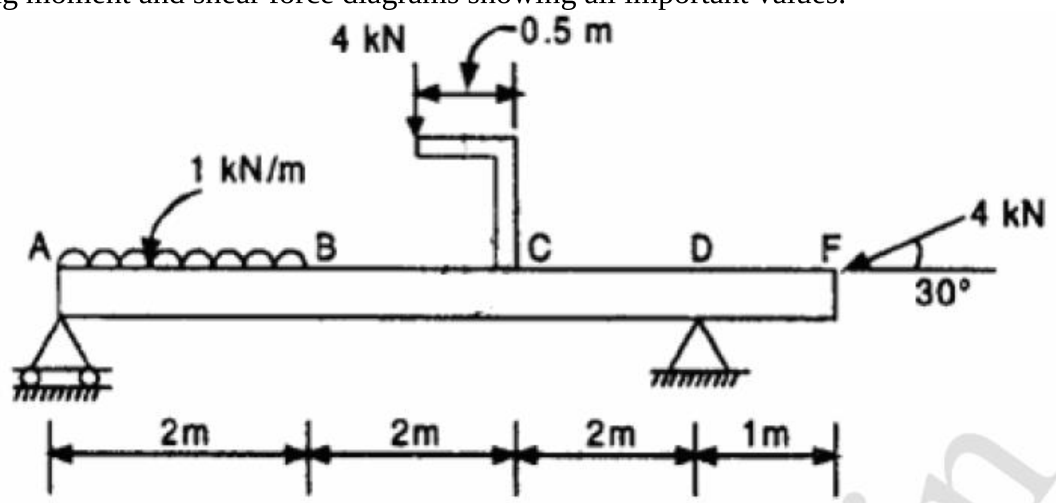
Question 8 A simply supported beam of length 'l' is subjected to a symmetrical uniformly varying load with zero intensity at the ends and intensity w (load per unit length) at the mid span. What is the maximum bending moment? Ans ($wl^2/12$)

Question 9 A cantilever beam of 2m length supports a triangularly distributed load over its entire length, the maximum of which is at the free end. The total load is 37.5 kN.What is the bending moment at the fixed end? Ans ($50x10^6$Nmm)

Question 10 A simply supported beam of length 10 m carries a uniformly varying load whose intensity varies from a maximum value of 5 kN/m at both ends to zero at the centre of the beam. It is desired to replace the beam with another simply supported beam which will be subjected to the same maximum 'bending moment' and 'shear force' as in the case of the previous one. Determine the length and rate of loading for the second beam if it is subjected to a uniformly distributed load over its whole length. Draw the variation of 'SF' and 'BM' in both the cases
(hints use sampled diagram below)

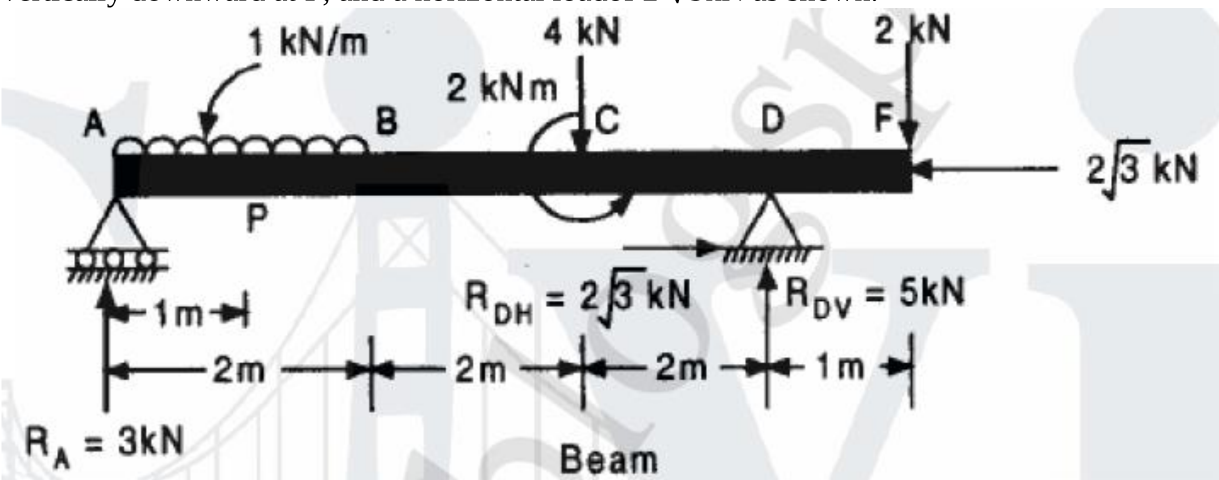


Question 11 Calculate the reactions at A and D for the beam shown in figure. Draw the bending moment and shear force diagrams showing all important values.

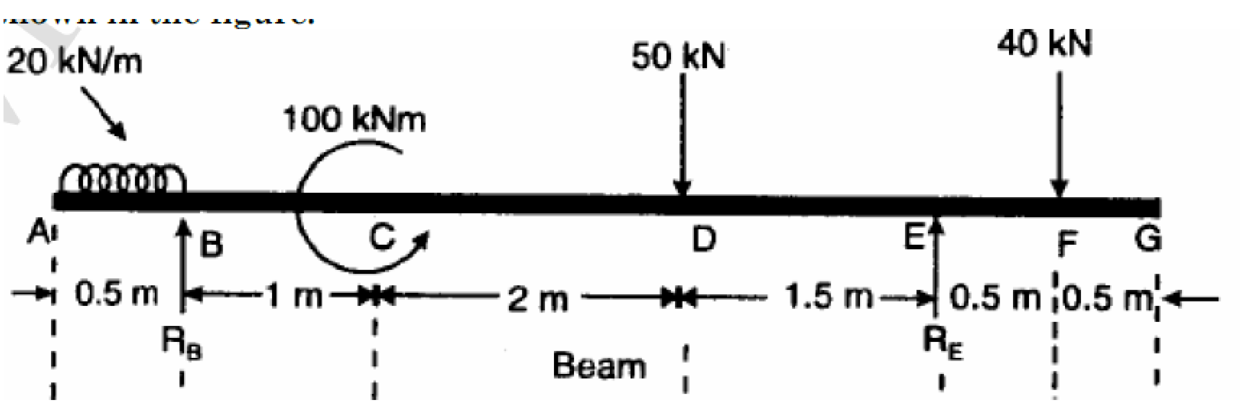


HINTS

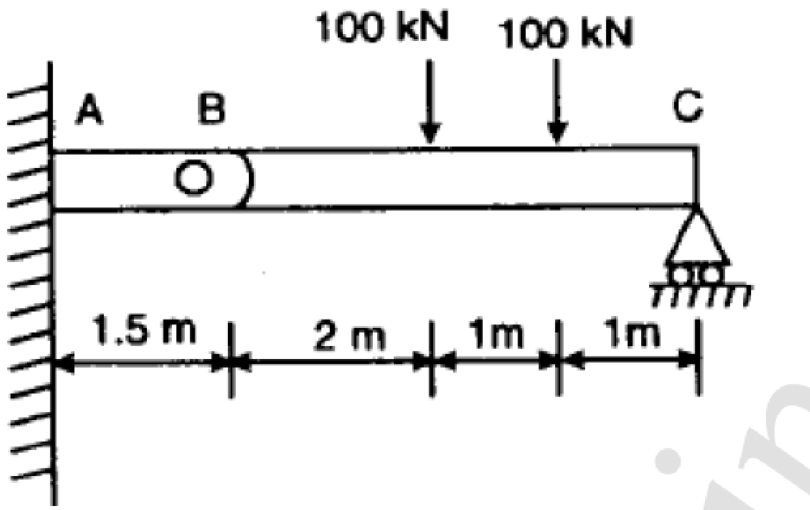
Equivalent figure below shows an overhanging beam ABCDF supported by a roller support at A and a hinged support at D. In the figure, a load of 4 kN is applied through a bracket 0.5 m away from the point C. Now apply equal and opposite load of 4 kN at C. This will be equivalent to a anticlockwise couple of the value of (4 x 0.5) = 2 kNm acting at C together with a vertical downward load of 4 kN at C. Show U.D.L. (1 kN/m) over the port AB, a point load of 2 kN vertically downward at F, and a horizontal loadof 2 $\sqrt{3}$kN as shown.



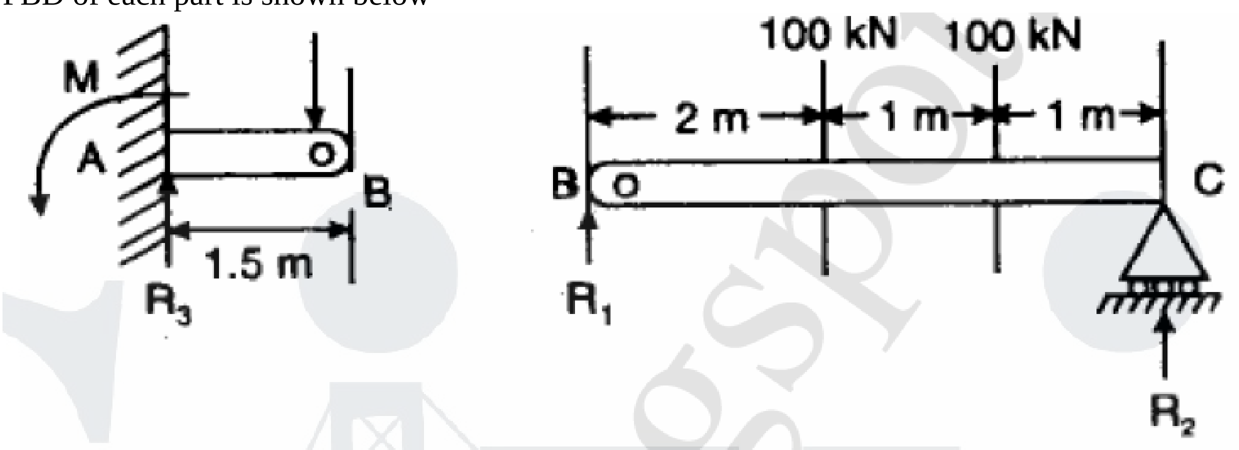
Question 12 Construct the bending moment and shearing force diagrams for the beam shown in the figure.



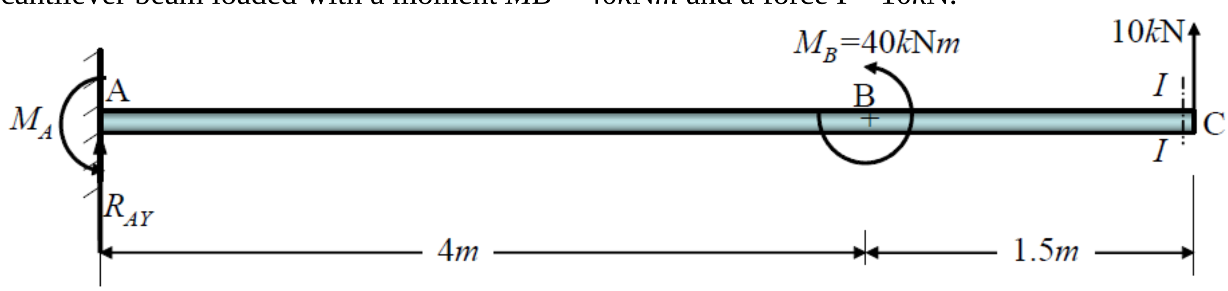
Question 13 Two bars AB and BC are connected by a frictionless hinge at B. The assembly is supported and loaded as shown in figure below. Draw the shear force and bending moment diagrams for the combined beam AC. clearly labelling the important values. Also indicate your sign convention.



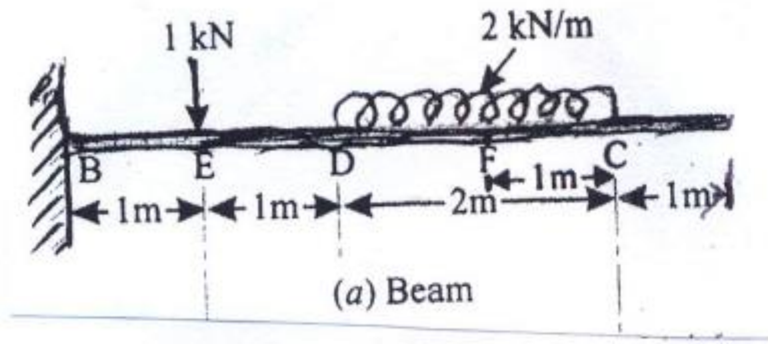
HINTS
There shall be a vertical reaction at hinge B and we can split the problem in two parts. Then the FBD of each part is shown below

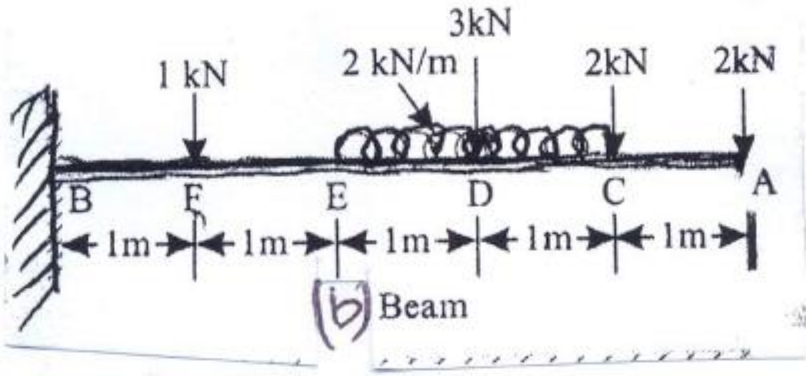


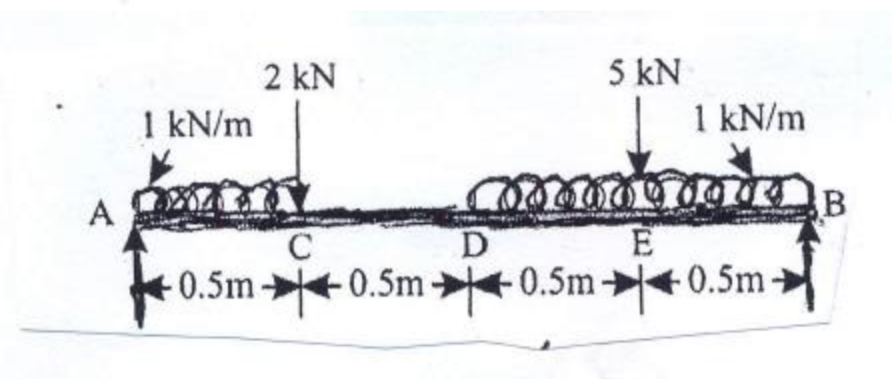
<u>Question 14</u> Determine the shear force and bending moment equations and plot them for a cantilever beam loaded with a moment *MB* = 40*k*N*m* and a force F= 10*k*N.



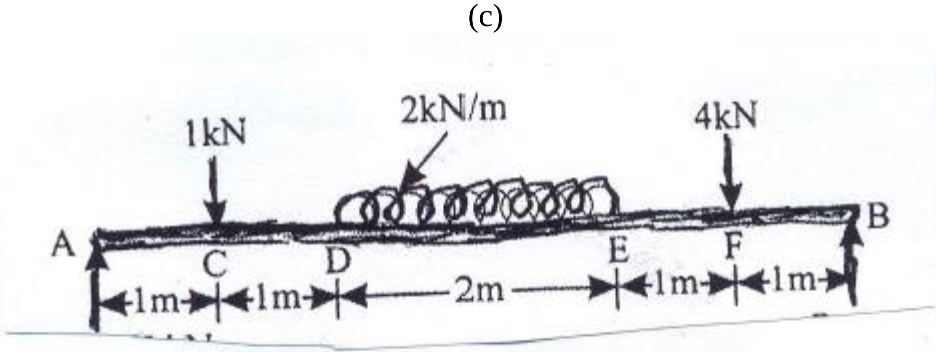
<u>Question 15:</u>
Draw the S.F and B.M of the loaded beam shown below in diagram (a) , (b),(c) and (d)
And clearly show the maximum bending moment and state its value at diagram (c) .


1 kN
2 kN/m
B
E
D
F
C
1m
1m
1m
2m
1m
(a) Beam


3kN
1 kN
2 kN/m
2kN
2kN
B
F
E
D
C
A
1m
1m
1m
1m
1m
(b) Beam


2 kN
5 kN
1 kN/m
1 kN/m
A
B
C
D
E
0.5m
0.5m
0.5m
0.5m

(c)


1kN
2kN/m
4kN
A
B
C
D
E
F
1m
1m
2m
1m
1m

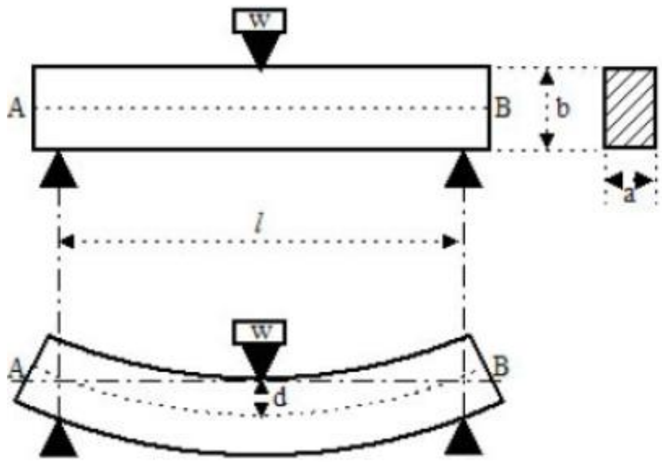
(d)

2. BENDING STRESS IN BEAMS

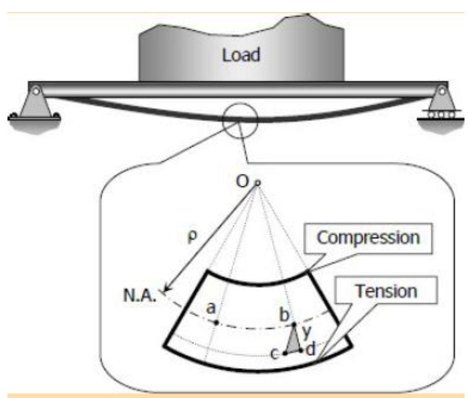
Stresses in Beams

Forces and couples acting on the beam cause bending (flexural stresses) and shearing stresses on any cross section of the beam and deflection perpendicular to the longitudinal axis of the beam. If couples are applied to the ends of the beam and no forces act on it, the bending is said to be pure bending. If forces produce the bending, the bending is called ordinary bending.

Any deformable bar subjected to a bending moment causes the material within the **bottom portion undergo tension** and the **top portion undergo compression**.



The region between the tension and compression lies the **neutral surface**, which the material do not undergo changes in length.



Several assumptions:

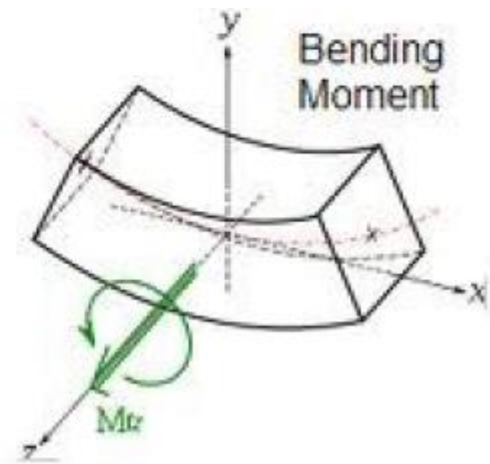
- The longitudinal axis, *x*, which lies within the **neutral surface** does not experience any **change in length**.
- All **cross sections** of the beam remain **plane** and perpendicular to the longitudinal axis during the deformation.
- Any **deformation** of the **cross section** within its own plane will be **neglected**.

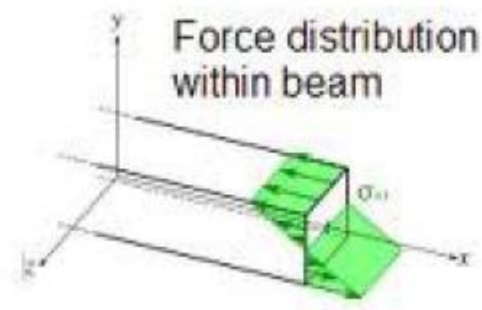
- The axis lying in the plane of the cross section and about which the cross section rotates is called the **neutral axis**.
- The material is **homogeneous**, with **same cross sectional area along the length** and **constant Elastic Modulus (E)**.

❖ Longitudinal strain varies linearly from zero at the neutral axis.

❖ Hooke's law applies when material is **homogeneous.**

❖ Neutral axis passes through the **centroid** of the cross-sectional area for linear-elastic material.





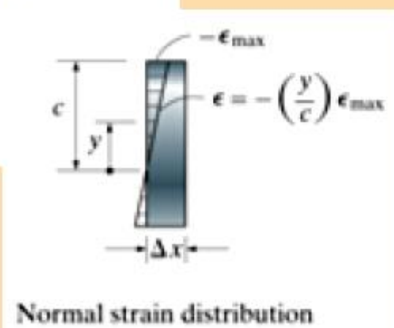
BENDING DEFORMATION

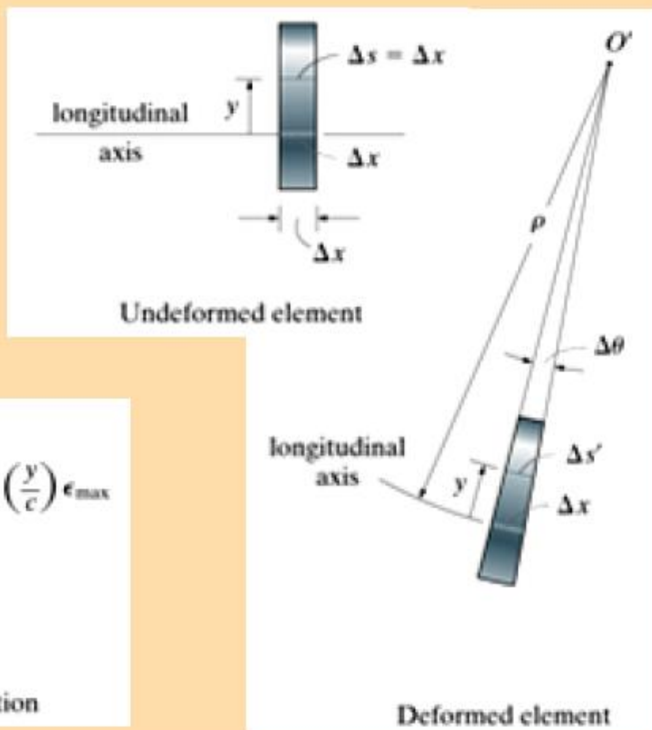
Normal Strain:

$$\varepsilon = \frac{\Delta s' - \Delta s}{\Delta s}$$

$$\varepsilon = \frac{(p-y)\Delta\theta - p\Delta\theta}{p\Delta\theta}$$

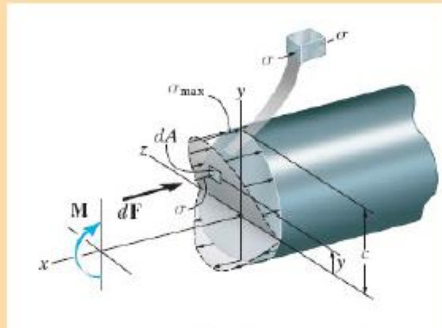
$$\therefore \varepsilon = -\frac{y}{p}$$





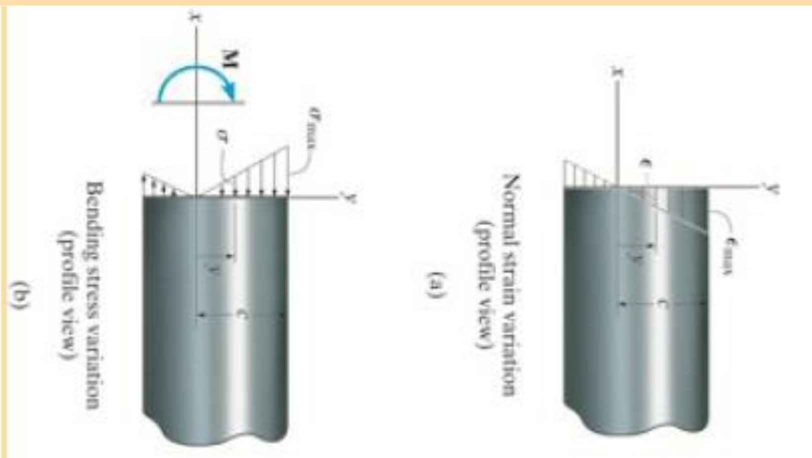
Resultant moment on the cross section is equal to the moment produced by the linear normal stress distribution about the neutral axis.

$$\sigma = \frac{My}{I}$$



Bending stress variation

σ	Normal stress in the member
M	Resultant internal moment
I	Moment of inertia
y	Perpendicular distance from the neutral axis

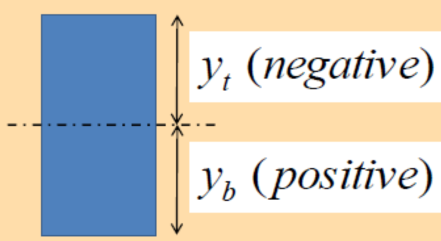


(b) Bending stress variation (profile view)

(a) Normal strain variation (profile view)

- Bending Stress, $\sigma = \frac{My}{I}$

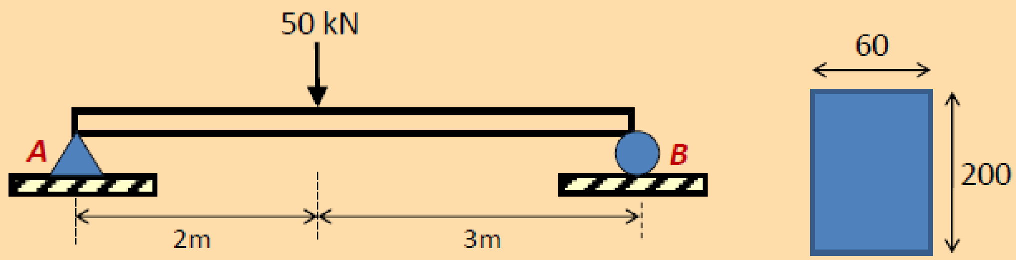
Sign Convention for y_t and y_b

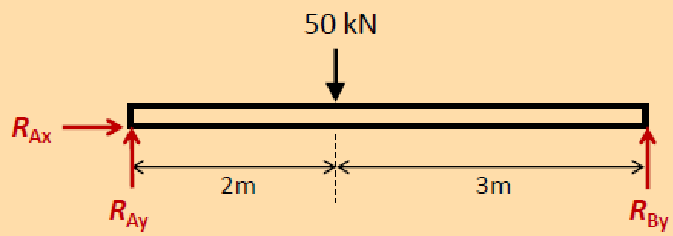


REMEMBER !! Also, moment may be positive or negative (sagging or hogging moment)

Example:01

Determine the maximum bending stress in beam AB as shown in the figure and draw the stress distribution over the cross section at this location.





By taking the moment at A:

$\Sigma M_A = 0$

$-R_B \times 5 + 50 \times 2 = 0$

$R_B = 20$ kN

$\Sigma F_y = 0$

$R_A + R_B = 50$

$R_A = 50 - 20$

$R_A = 30$ kN

$\Sigma F_x = 0$

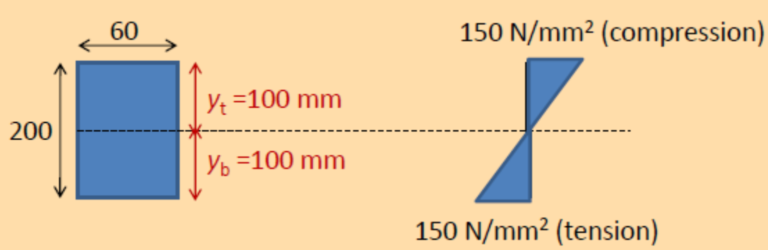
$R_{Ax} = 0$

$M_{max} = 60$ kNm (occurred at 2 m from A)

$I = bh^3/12 = 60 \times 200^3 / 12 = 40 \times 10^6$ mm^4

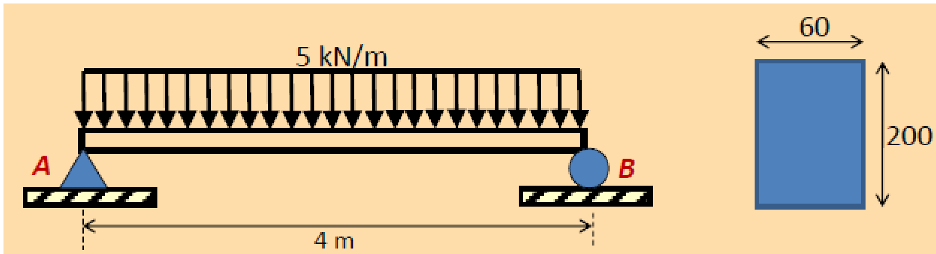
$y_t = y_b = 100$ mm

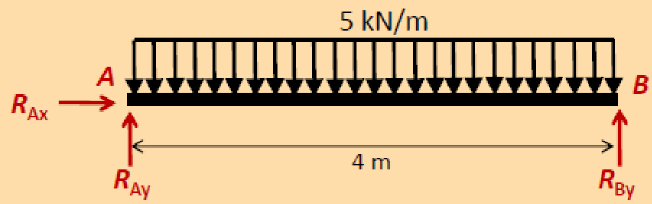
$\sigma_{max} = My / I = (60 \times 10^6 \times 100) / (40 \times 10^6) = 150$ N/mm^2 (+ve & -ve)



Example:02

Determine the maximum bending stress in beam AB as shown in the figure and draw the stress distribution over the cross section at this location.





By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times 4 + 5 \times 4 \times 4/2 = 0$

$R_{By} = 10$ kN

$\Sigma F_y = 0$

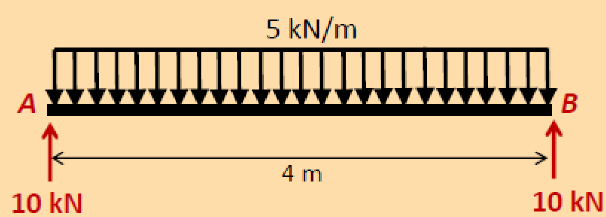
$R_{Ay} + R_{By} = 5 \times 4$

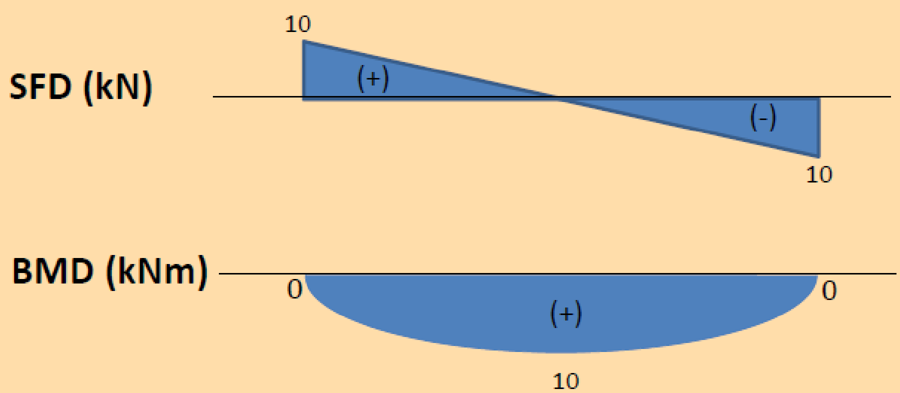
$R_{Ay} = 20 - 10$

$R_{Ay} = 10$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$



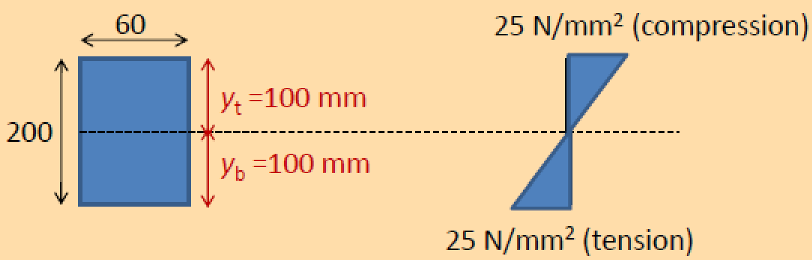


M_{max} = 10 kNm (occurred at 2 m from A)

$I = bh^3/12 = 60 \times 200^3 / 12 = 40 \times 10^6$ mm^4

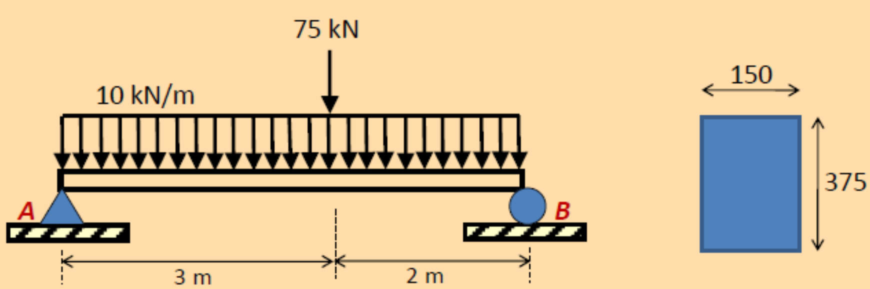
$y_t = y_b = 100$ mm

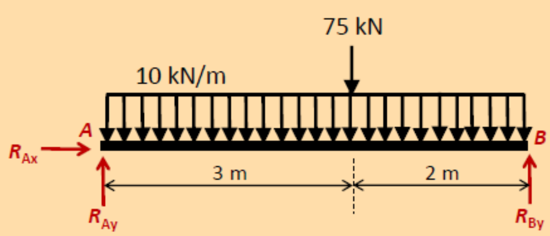
$\sigma_{max} = My / I = (10 \times 10^6 \times 100) / (40 \times 10^6) = 25$ N/mm^2



Example:03

Determine the maximum bending stress in beam AB as shown in the figure and draw the stress distribution over the cross section at this location.





By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times 5 + 10 \times 5 \times 5/2 + 75 \times 3 = 0$

$R_{By} = 70$ kN

$\Sigma F_y = 0$

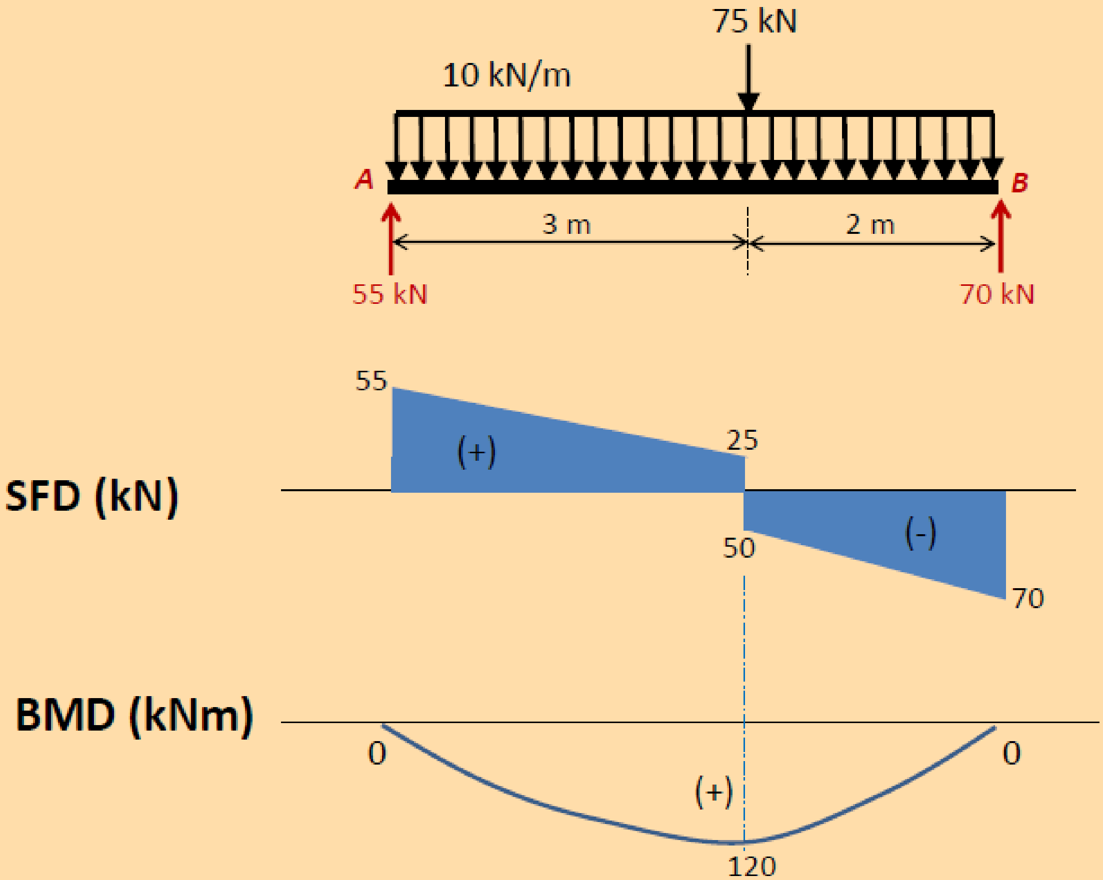
$R_{Ay} + R_{By} = 10 \times 5 + 75$

$R_{Ay} = 125 - 70$

$R_{Ay} = 55$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$

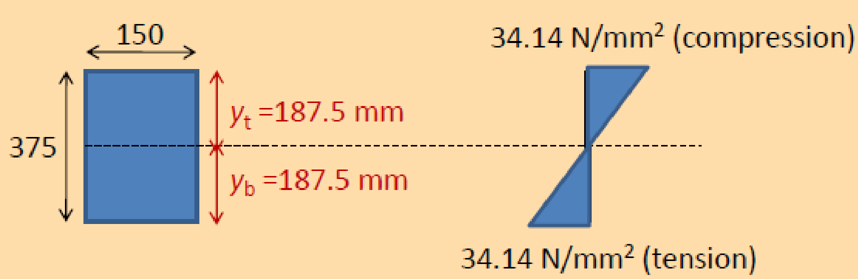


M_{max} = 120 kNm (occurred at 3 m from A)

$I = bh^3/12 = 150 \times 375^3 / 12 = 6.59 \times 10^8$ mm^4

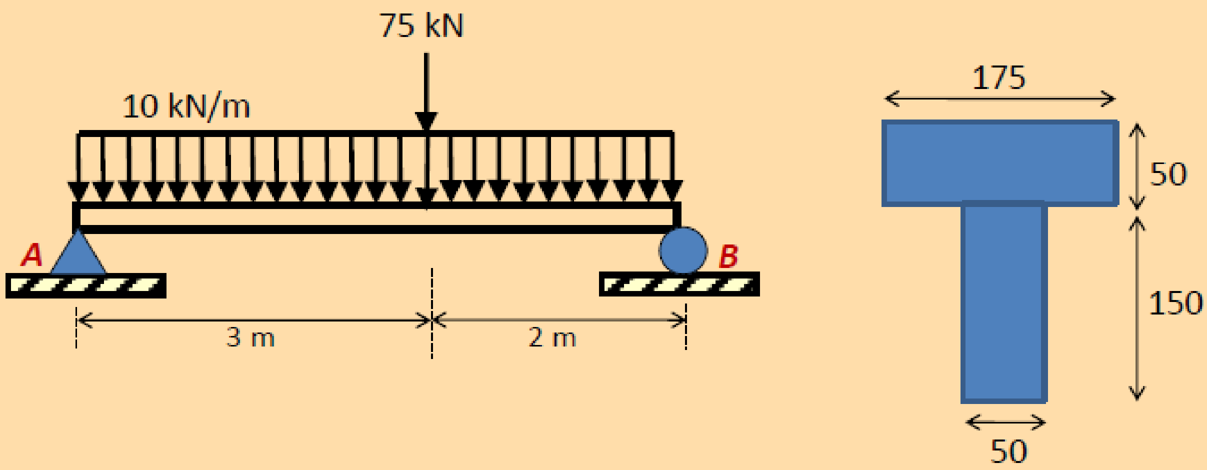
$y_t = y_b = 187.5$ mm

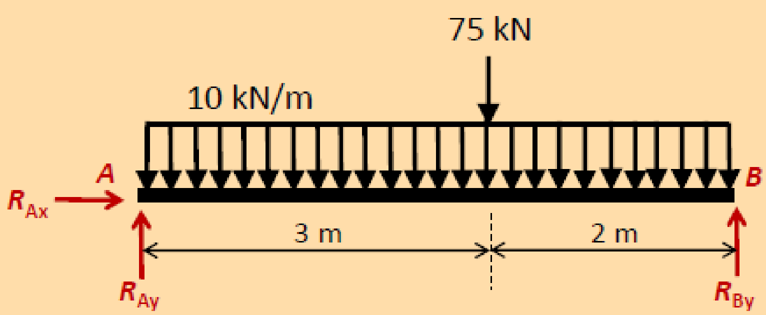
$\sigma_{max} = My / I = (120 \times 10^6 \times 187.5) / (6.59 \times 10^8) = 34.14$ N/mm^2



Example:04

Determine the maximum bending stress in beam AB as shown in the figure and draw the stress distribution over the cross section at this location.





By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times 5 + 10 \times 5 \times 5/2 + 75 \times 3 = 0$

$R_{By} = 70$ kN

$\Sigma F_y = 0$

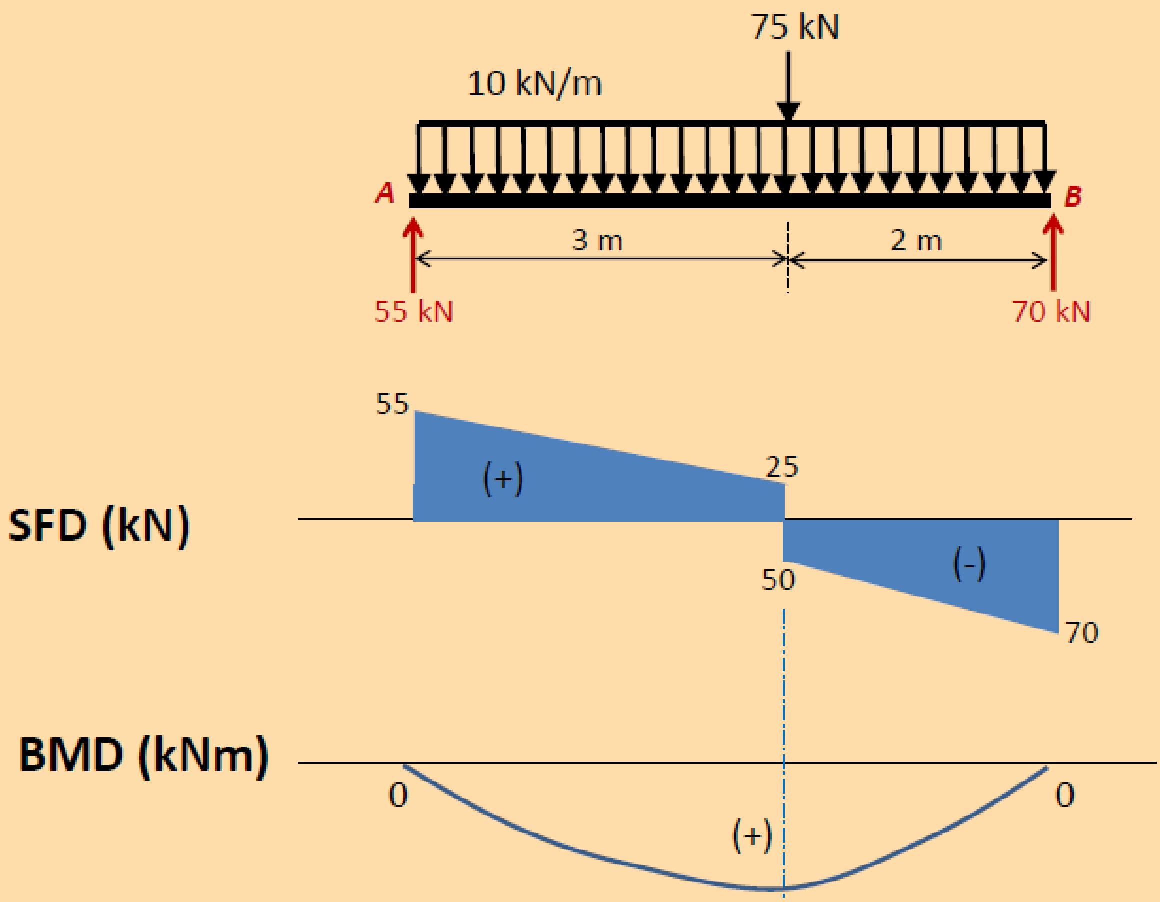
$R_{Ay} + R_{By} = 10 \times 5 + 75$

$R_{Ay} = 125 - 70$

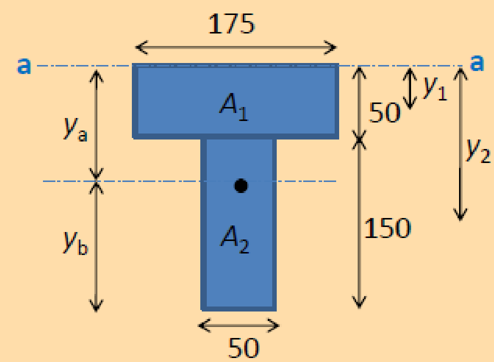
$R_{Ay} = 55$ kN

$\Sigma F_x = 0$

$R_{Ax} = 0$



Determine the centroid of the section:



Area 1: $175 \times 50 = 8750\ \text{mm}^2$

Area 2: $150 \times 50 = 7500\ \text{mm}^2$

Taking moment at a – a:

$\Sigma A \cdot y_a = A_1 \cdot y_1 + A_2 \cdot y_2$

$\Sigma A \cdot y_a = 8750 \times 25 + 7500 \times 125$

$y_a = 1156250 / (8750 + 7500) = 71.15\ \text{mm}$

$y_b = (150 + 50) - 71.15 = 128.85\ \text{mm}$

Determine the Second Moment of Area, I:

$I = \Sigma bh^3/12 + Ay^2$

$= 175 \times 50^3 / 12 + 8750 \times 46.15^2 + 50 \times 150^3 / 12 + 7500 \times 53.85^2$

$= 20.46 \times 10^6 + 35.81 \times 10^6 \text{ mm}^4$

$= 56.27 \times 10^6 \text{ mm}^4$

M_{max} = 120 kNm (occurred at 3 m from A)

$\sigma_{max\ (top)} = My / I$

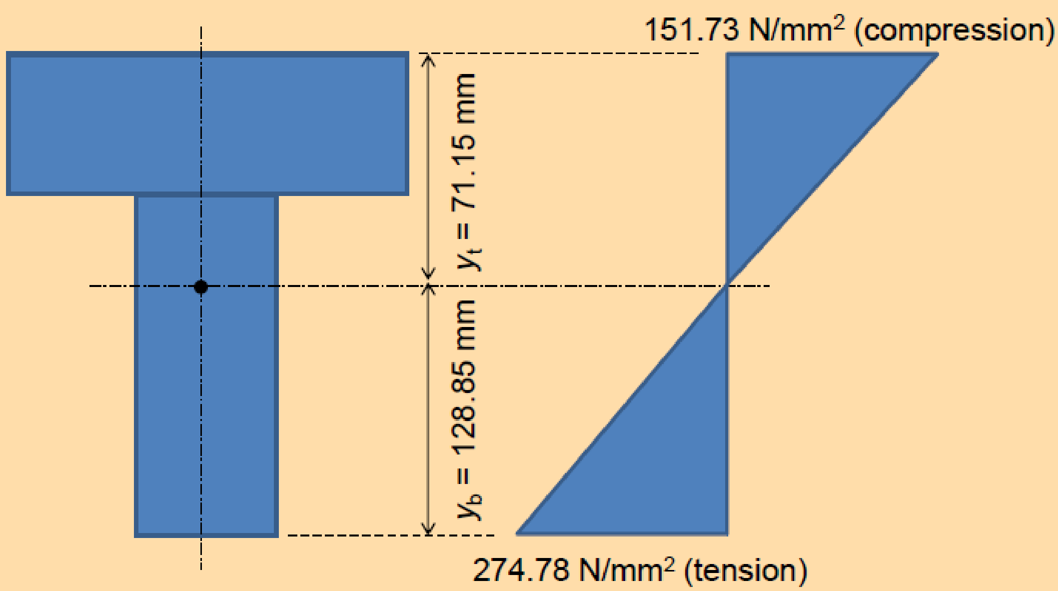
$= (120 \times 10^6 \times -71.15) / (56.27 \times 10^6)$

$= -151.73 \text{ N/mm}^2$

$\sigma_{max\ (bottom)} = My / I$

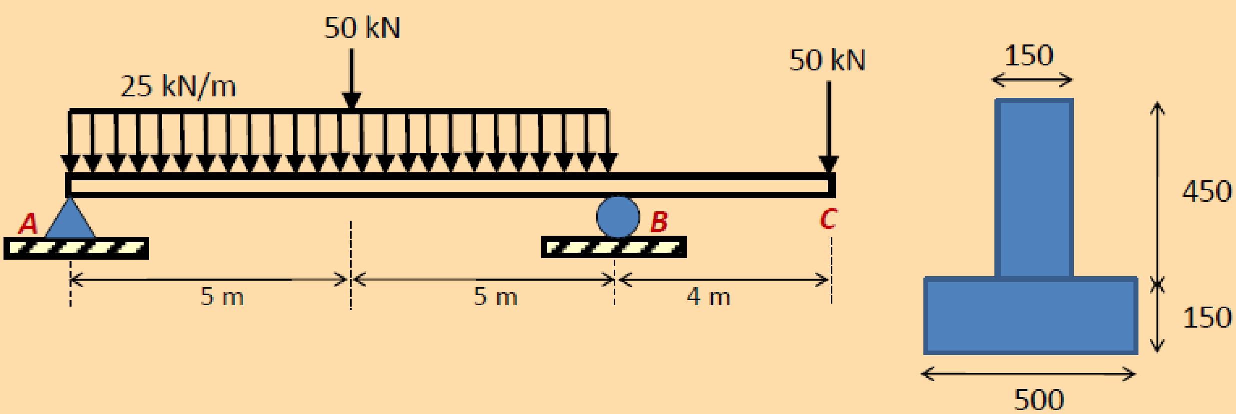
$= (120 \times 10^6 \times 128.85) / (56.27 \times 10^6)$

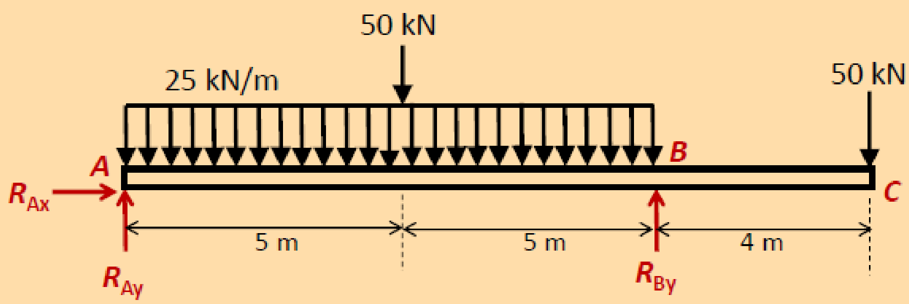
$= 274.78 \text{ N/mm}^2$



Example:05

Determine the maximum compressive and tensile stress in beam AB as shown in the figure and draw the stress distribution over the cross section at this location.





By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times 10 + 25 \times 10 \times 10/2 + 50 \times 5 + 50 \times 14 = 0$

$R_{By} = 220$ kN

$\Sigma F_y = 0$

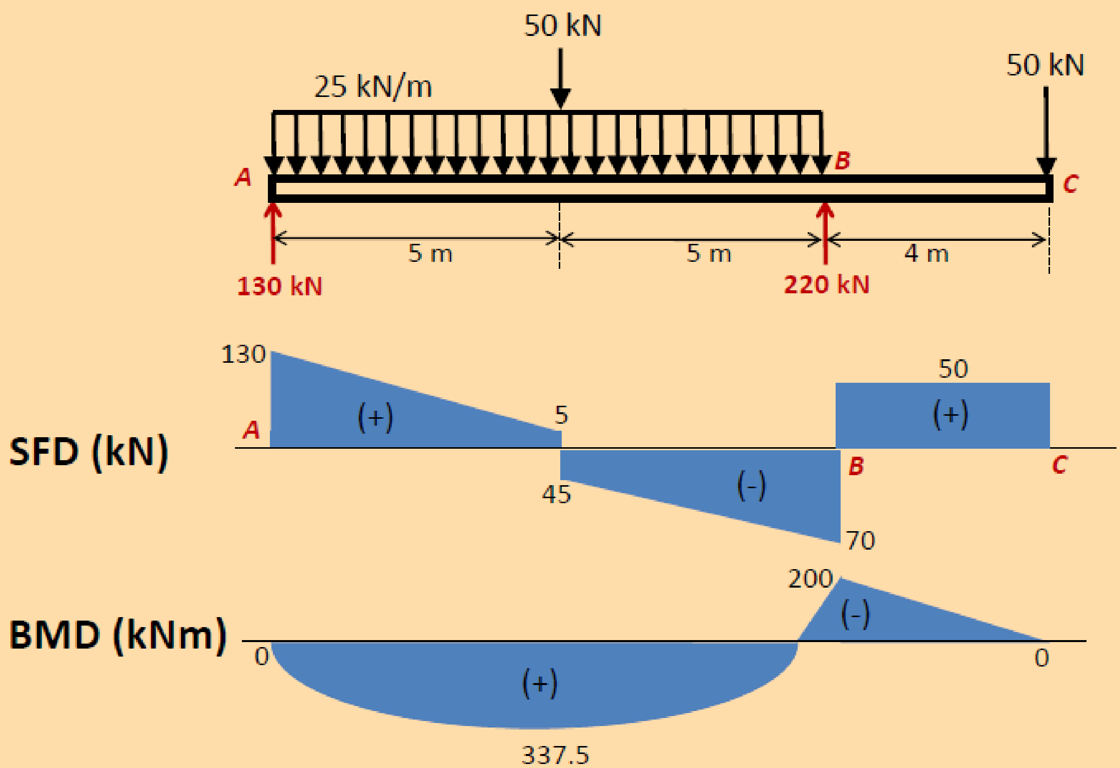
$R_{Ay} + R_{By} = 25 \times 10 + 50 + 50$

$R_{Ay} = 350 - 220$

$R_{Ay} = 130$ kN

$\Sigma F_x = 0$

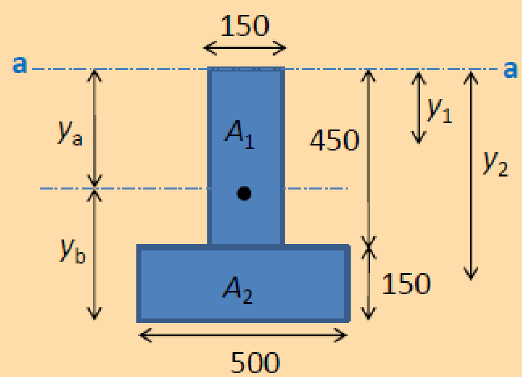
$R_{Ax} = 0$



Determine the centroid of the section:

Area 1: $450 \times 150 = 67500 \text{ mm}^2$

Area 2: $150 \times 500 = 75000 \text{ mm}^2$



Taking moment at a–a:

$\Sigma A \cdot y_a = A_1 \cdot y_1 + A_2 \cdot y_2$

$\Sigma A \cdot y_a = 67500 \times 225 + 75000 \times 525$

$y_a = 54562500 / (67500 + 75000) = 382.89 \text{ mm}$

$y_b = (450 + 150) - 382.89 = 217.11 \text{ mm}$

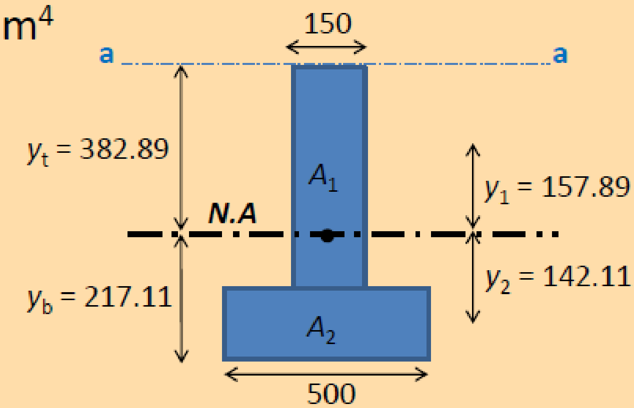
Determine the Second Moment of Area, I:

$I = \Sigma bh^3/12 + Ay^2$

$= 150 \times 450^3 / 12 + 67500 \times 157.89^2 + 500 \times 150^3 / 12 + 75000 \times 142.11^2$

$= 28.22 \times 10^8 + 16.55 \times 10^8 \text{ mm}^4$

$= 44.77 \times 10^8 \text{ mm}^4$



M_{max} = 335 kNm (occurred at 5 m from A): **Sagging moment**

$\sigma_{max\ (top)} = My / I$

$= (335 \times 10^6 \times -382.89) / (44.77 \times 10^8) = -28.65 \text{ N/mm}^2$

$\sigma_{max\ (bottom)} = My / I$

$= (335 \times 10^6 \times 217.11) / (44.77 \times 10^8) = 16.24 \text{ N/mm}^2$

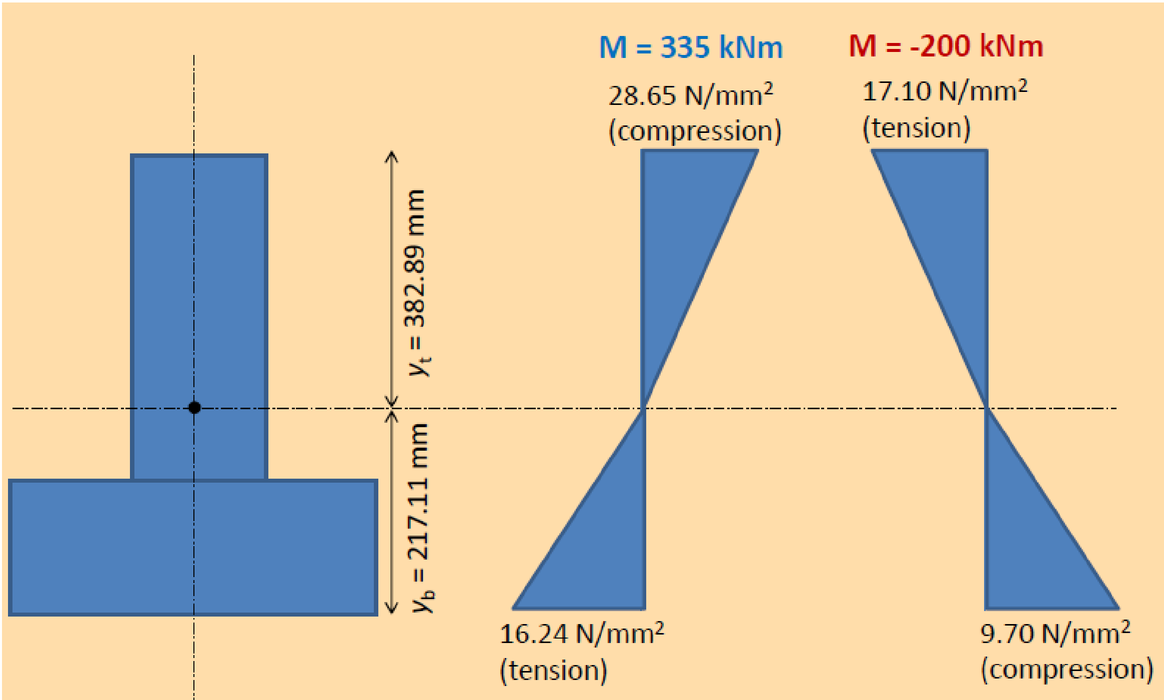
M_{max} = -200 kNm (occurred at support B): **Hogging moment**

$\sigma_{max\ (top)} = My / I$

$= (-200 \times 10^6 \times -382.89) / (44.77 \times 10^8) = 17.10 \text{ N/mm}^2$

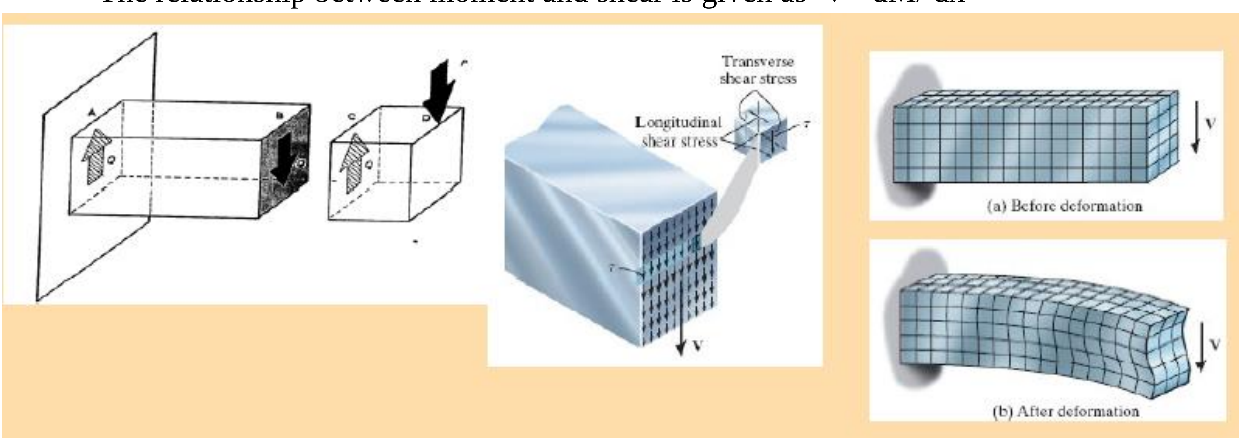
$\sigma_{max\ (bottom)} = My / I$

$= (-200 \times 10^6 \times 217.11) / (44.77 \times 10^8) = -9.70 \text{ N/mm}^2$

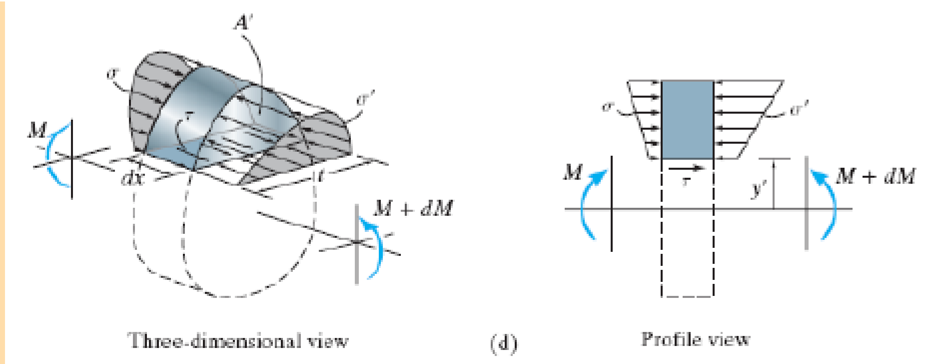


SHEAR IN STRAIGHT MEMBERS

- When a shear **V** is applied, non-uniform shear-strain distribution over the cross section will cause the cross section to *warp*.
- The relationship between moment and shear is given as V= dM/ dx



The shear formula is used to find the transverse shear stress on the beam's cross-sectional area.



Three-dimensional view (d) Profile view

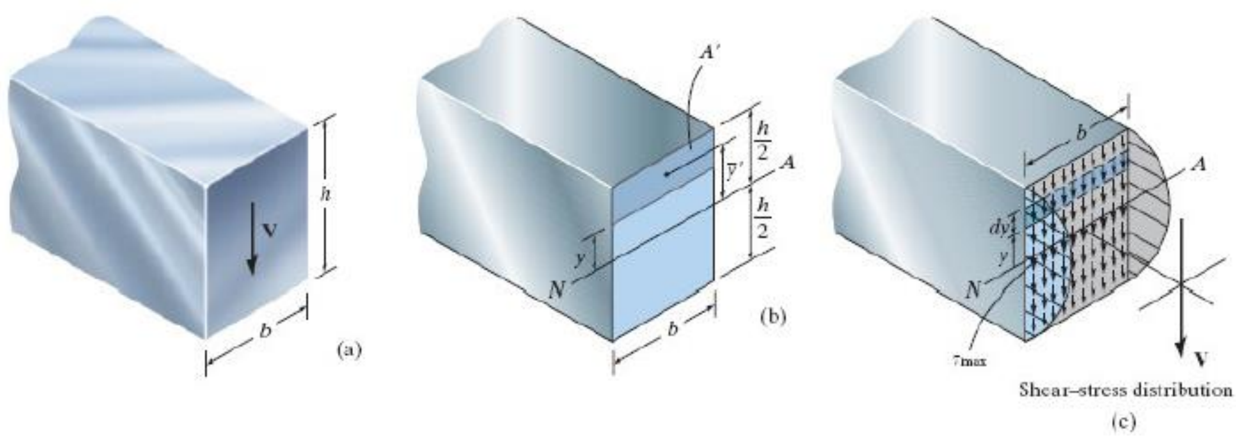
$$\tau = \frac{VQ}{It}$$

$$\text{where } Q = \int_{A'} y \cdot dA = \bar{y}' \cdot A'$$

τ	The shear stress in the member
V	Internal resultant shear force
I	Moment of inertia of the *entire* cross-sectional area
t	Width of the member's cross-sectional area

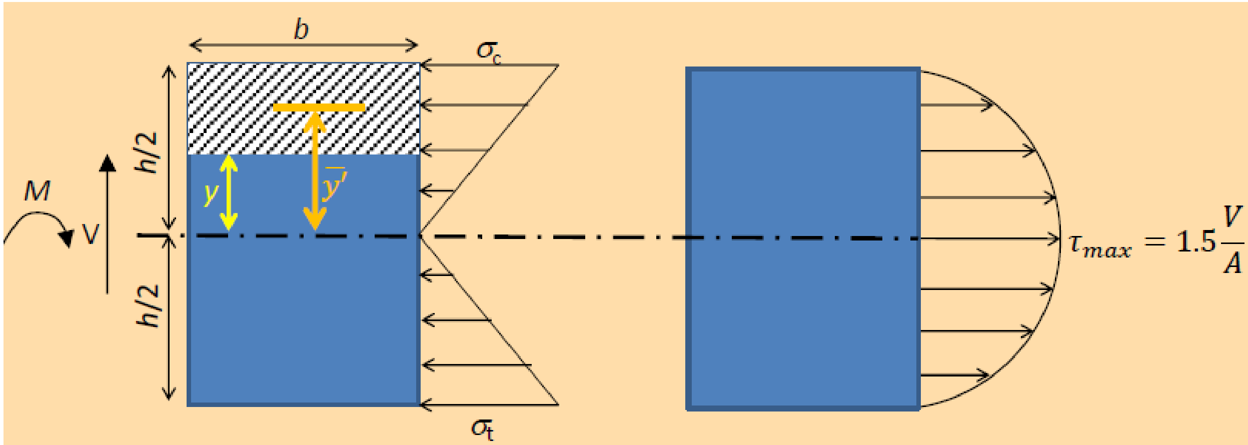
SHEAR STRESSES IN BEAMS

For rectangular cross section, **shear stress varies parabolically** with depth and maximum shear stress is along the neutral axis.



(a) (b) Shear–stress distribution (c)

$$\tau_{max} = 1.5\frac{V}{A}$$



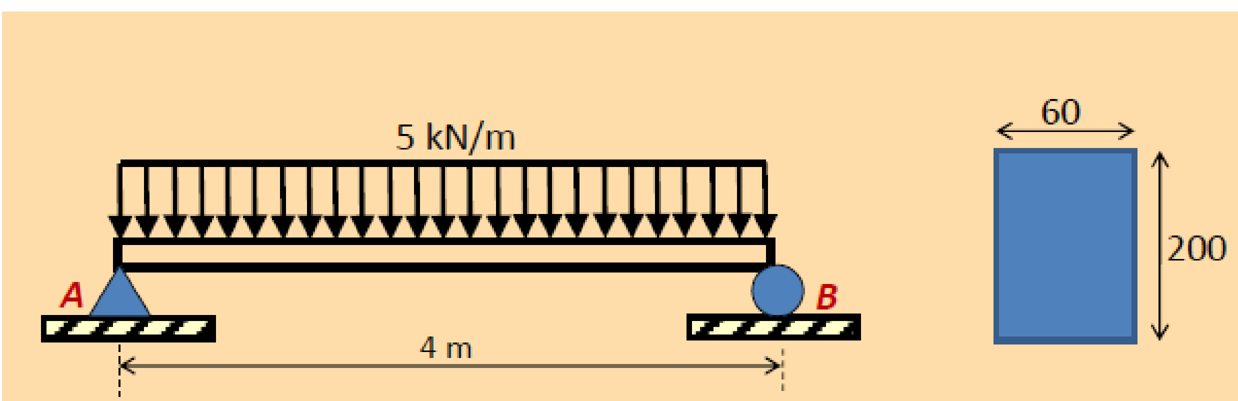
$$Q = \int_{A'} y \cdot dA = \bar{y}' \cdot A' = \left[y + \frac{1}{2}\left(\frac{h}{2} - y\right)\right] \times \left[\left(\frac{h}{2} - y\right) b\right] \qquad and\ I = \frac{bh^3}{12}$$

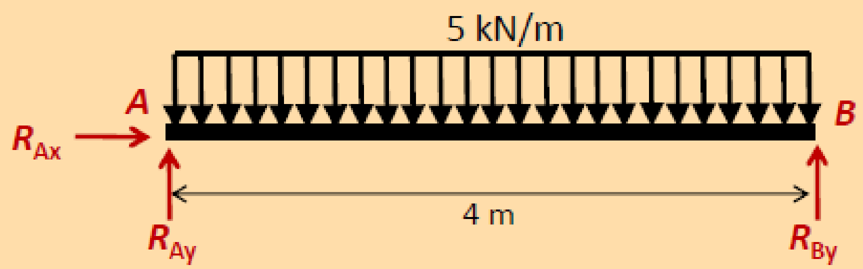
$$Therefore, \tau = \frac{VQ}{It} = \frac{V\left(\frac{1}{2}\right)\left[\frac{h^2}{4} - y^2\right] b}{\left(\frac{bh^3}{12}\right) b} = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right)$$ ➔ **Parabolic Eq.**

$$Therefore\ at\ y = 0, \tau_{max} = 1.5\frac{V}{A}$$

Example:06

Determine the maximum shear stress in beam AB at 1 m from A and draw the stress distribution over the cross section at every 25 mm interval.





By taking the moment at A:

$\Sigma M_A = 0$

$-R_{By} \times 4 + 5 \times 4 \times 4/2 = 0$

$R_{By} = 10$ kN

$\Sigma F_y = 0$

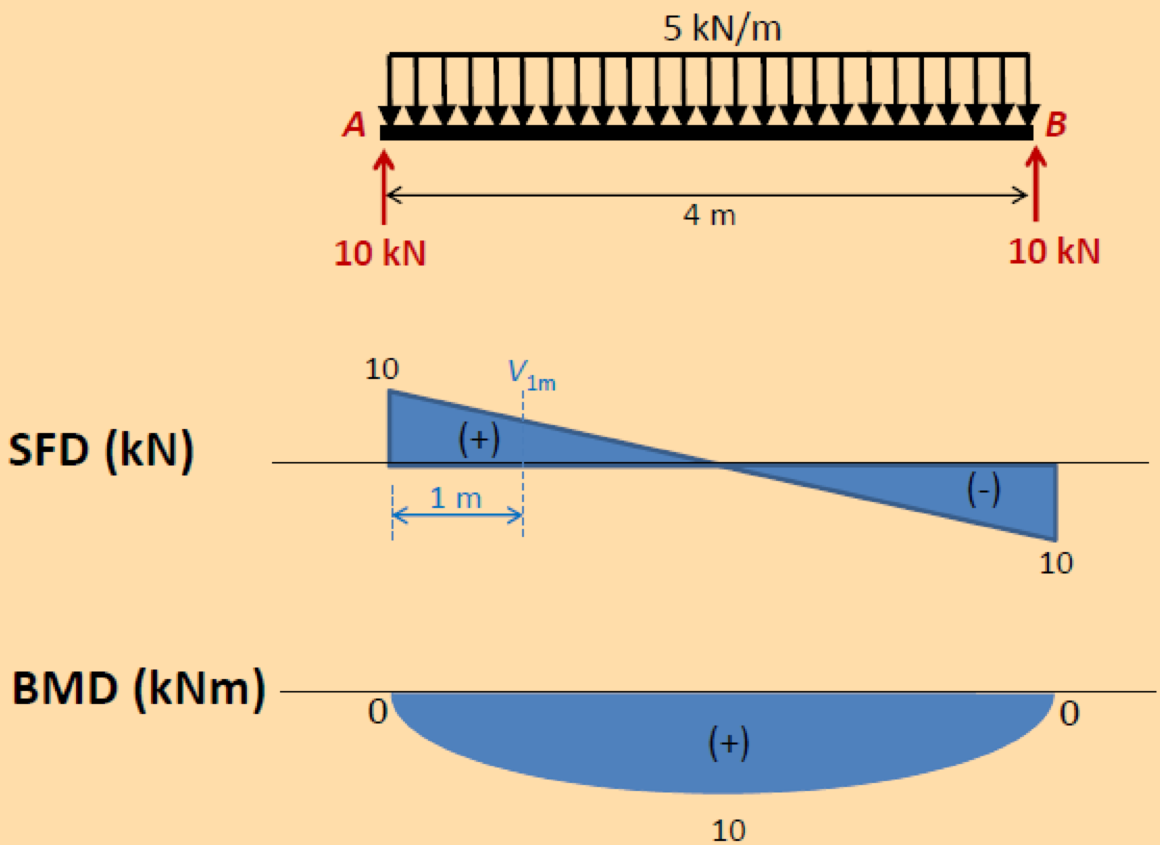
$R_{Ay} + R_{By} = 5 \times 4$

$R_{Ay} = 20 - 10$

$R_{Ay} = 10$ kN

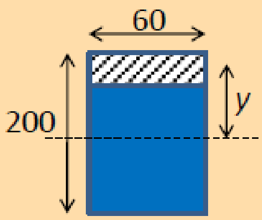
$\Sigma F_x = 0$

$R_{Ax} = 0$



$\therefore V_{1m} = 5\ \text{kN}$

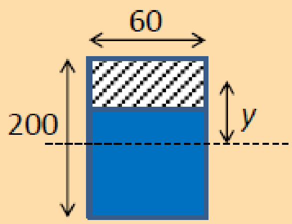
$I = bh^3/12 = 60 \times 200^3 / 12 = 40 \times 10^6\ \text{mm}^4$



$A = 60 \times 25 = 1500\ \text{mm}^2$

$y = 100 - (25/2) = 87.5\ \text{mm}$

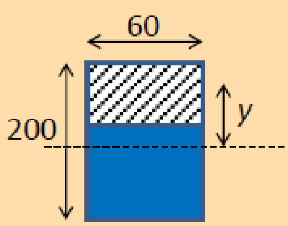
$$\tau_1 = \frac{VAy}{It} = \frac{5\times10^3 \times 1500 \times 87.5}{40\times10^6 \times 60} = 0.273\text{N/mm}^2$$



$A = 60 \times 50 = 3000\ \text{mm}^2$

$y = 100 - (50/2) = 75\ \text{mm}$

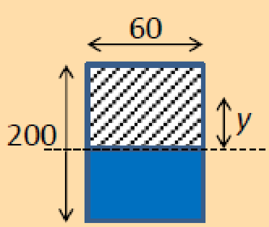
$$\tau_2 = \frac{VAy}{It} = \frac{5\times10^3 \times 3000 \times 75}{40\times10^6 \times 60} = 0.469\text{N/mm}^2$$



$A = 60 \times 75 = 4500\ \text{mm}^2$

$y = 100 - (75/2) = 62.5\ \text{mm}$

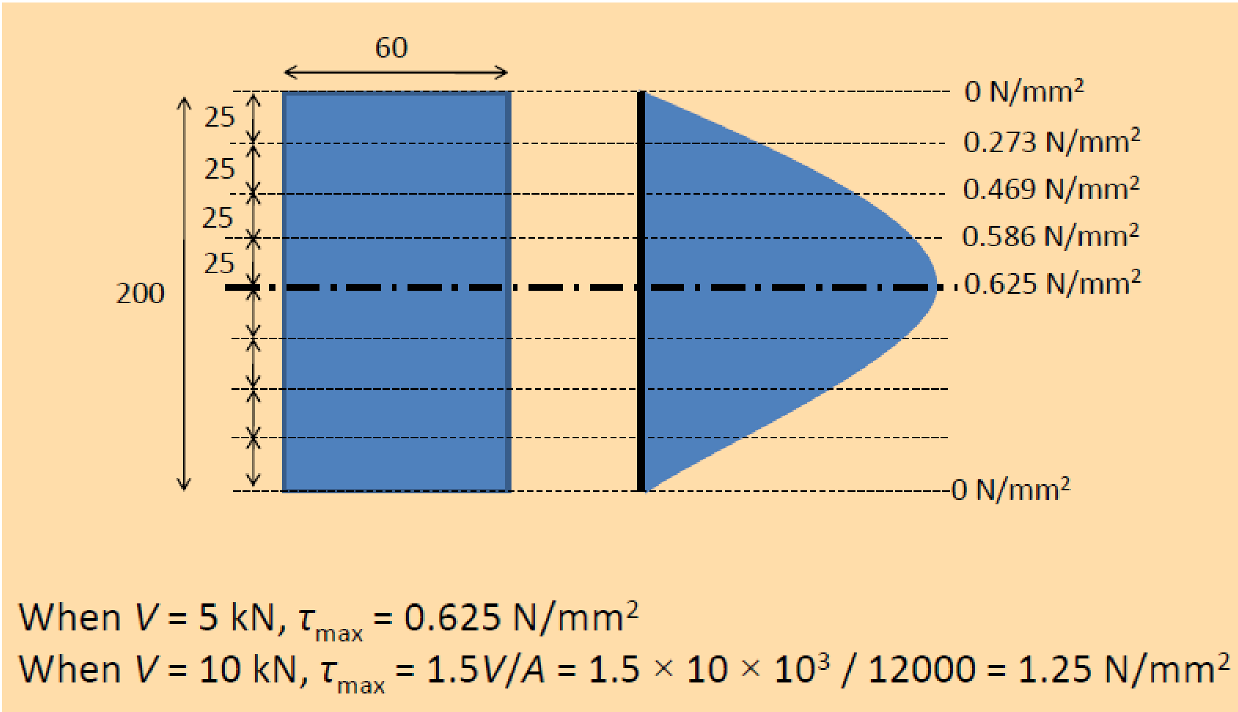
$$\tau_3 = \frac{VAy}{It} = \frac{5\times10^3 \times 4500 \times 62.5}{40\times10^6 \times 60} = 0.586\text{N/mm}^2$$



$A = 60 \times 100 = 6000\ \text{mm}^2$

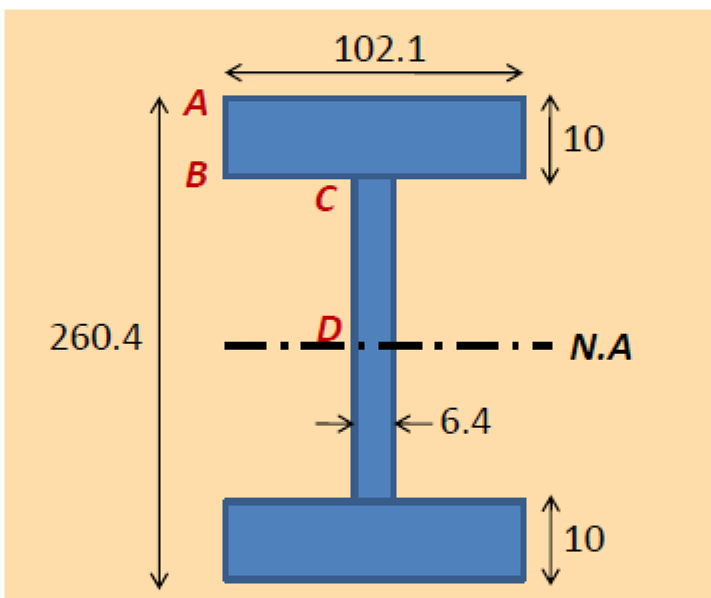
$y = 100 - (100/2) = 50\ \text{mm}$

$$\tau_4 = \frac{VAy}{It} = \frac{5\times10^3 \times 6000 \times 50}{40\times10^6 \times 60} = 0.625\text{N/mm}^2$$

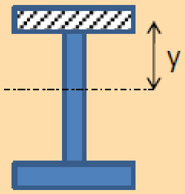


Example:07

Determine the shear stress distribution over the cross section at ***A***, ***B***, ***C*** and ***D*** as shown in the figure. Given $V = 25$ kN and $I = 4008$ cm^4.



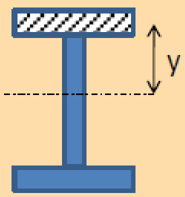
$$\tau_A = 0$$



$A = 102.1 \times 10 = 1021\ \text{mm}^2$

$y = (260.4/2) - (10/2) = 125.2\ \text{mm}$

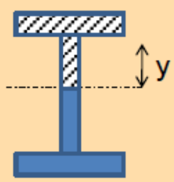
$$\tau_B = \frac{VAy}{It} = \frac{25\times10^3 \times 1021 \times 125.2}{40.08\times10^6 \times 102.1} = 0.781\text{N/mm}^2$$



$A = 102.1 \times 10 = 1021\ \text{mm}^2$

$y = (260.4/2) - (10/2) = 125.2\ \text{mm}$

$$\tau_C = \frac{VAy}{It} = \frac{25\times10^3 \times 1021 \times 125.2}{40.08\times10^6 \times 6.4} = 12.458\text{N/mm}^2$$



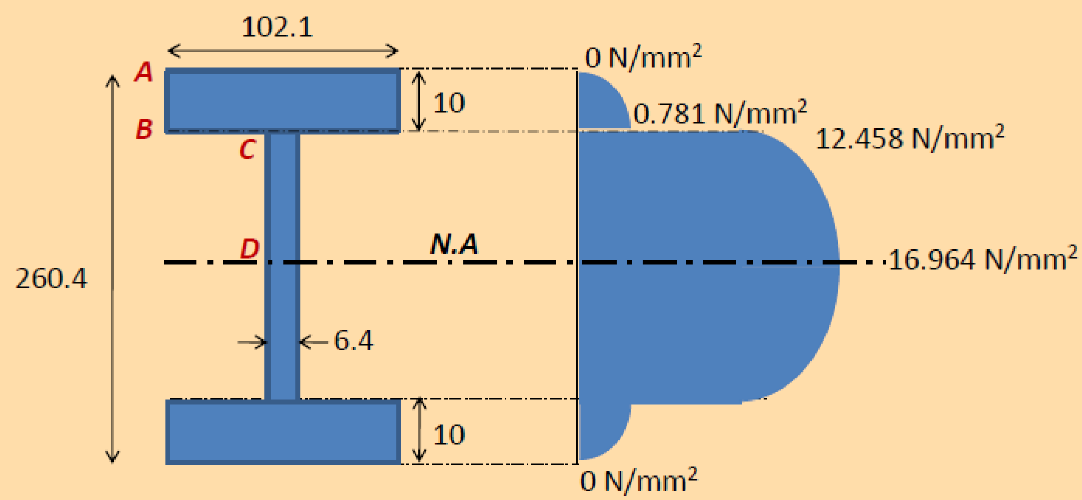
$A_1 = 102.1 \times 10 = 1021\ \text{mm}^2$

$y_1 = (260.4/2 - 10/2) = 125.2\ \text{mm}$

$A_2 = (260.4/2 - 10) \times 6.4 = 769.28\ \text{mm}^2$

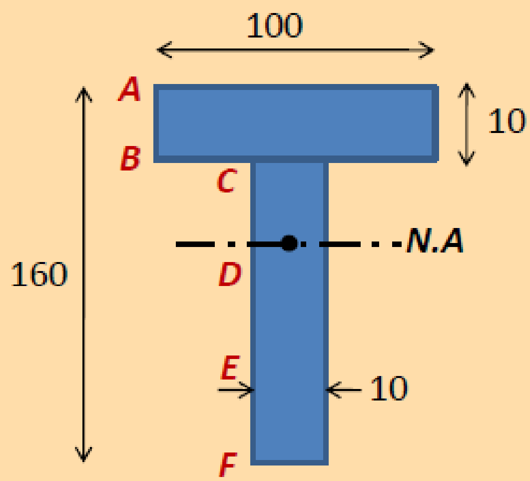
$y_2 = (260.4/2 - 10)/2 = 60.1\ \text{mm}$

$$\tau_D = \frac{V\sum A_i y_i}{It} = \frac{25\times10^3 \times (1021\times125.2 + 769.28\times60.1)}{40.08\times10^6 \times 6.4} = 16.964\text{N/mm}^2$$



Example:08

Determine the shear stress distribution over the cross section at ***A***, ***B***, ***C***, ***D***, ***E*** and ***F*** as shown in the figure. Given V = 25 kN.



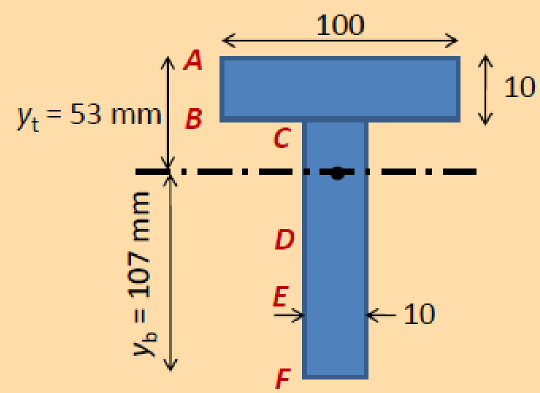
Determine the centroid of the section:

Total cross sectional area, A = 100 × 10 + 150 × 10 = 2500 mm^2

$$y_t = \frac{(100\times10)(10/2)+(10\times150)(150/2+10)}{2500}$$

$$y_t = 53mm$$

$$y_b = 160 - 53 = 107mm$$



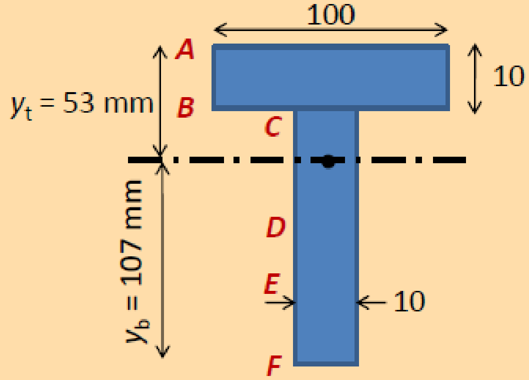
Determine the Second Moment of Area:

$$I = bh^3/12 + Ay$$

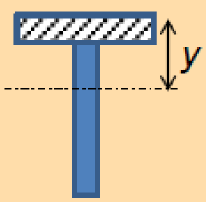
$$I = 100\times10^3/12 + 100\times10\times(53-5)^2$$

$$+10\times150^3/12 + 10\times150\times(150/2+10-53)^2$$

$$I = 2.31\times10^6 + 4.35\times10^6 = 6.66\times10^6\,mm^4$$



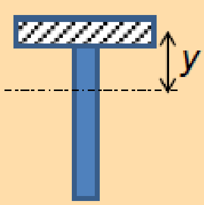
$$\tau_A = \tau_F = 0$$



A = 100 × 10 = 1000 mm^2

y = (53 – 10/2) = 48 mm

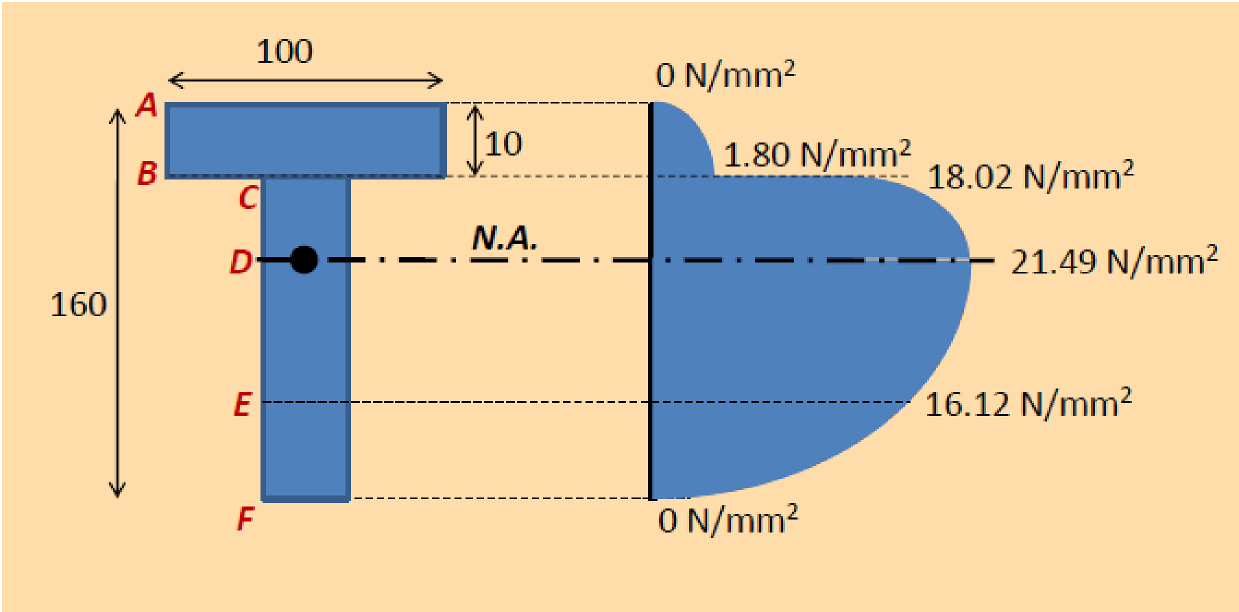
$$\tau_B = \frac{VAy}{It} = \frac{25\times10^3\times100\times10\times48}{6.66\times10^6\times100} = 1.80\text{N/mm}^2$$



A = 100 × 10 = 1000 mm^2

y = (53 – 10/2) = 48 mm

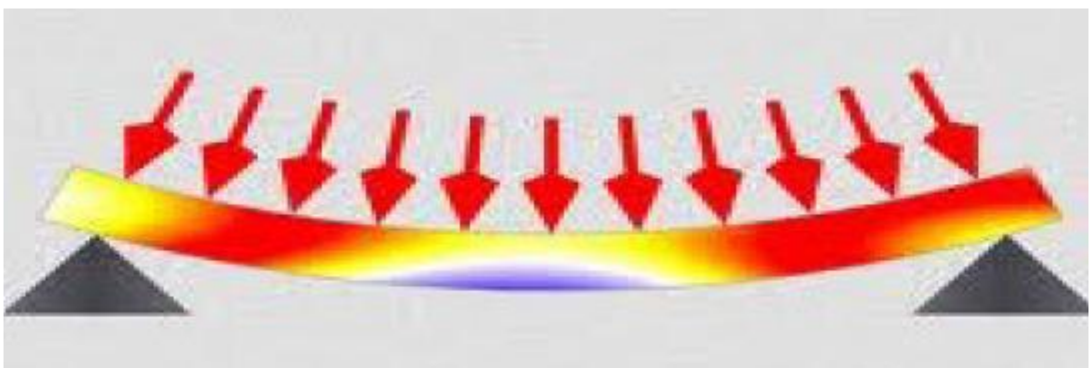
$$\tau_C = \frac{VAy}{It} = \frac{25\times10^3\times100\times10\times48}{6.66\times10^6\times10} = 18.02\text{N/mm}^2$$



3. <u>DEFLECTION OF BEAMS</u>

INTRODUCTION

Deflection is a result from the load action to the beam (self weight, service load etc.)

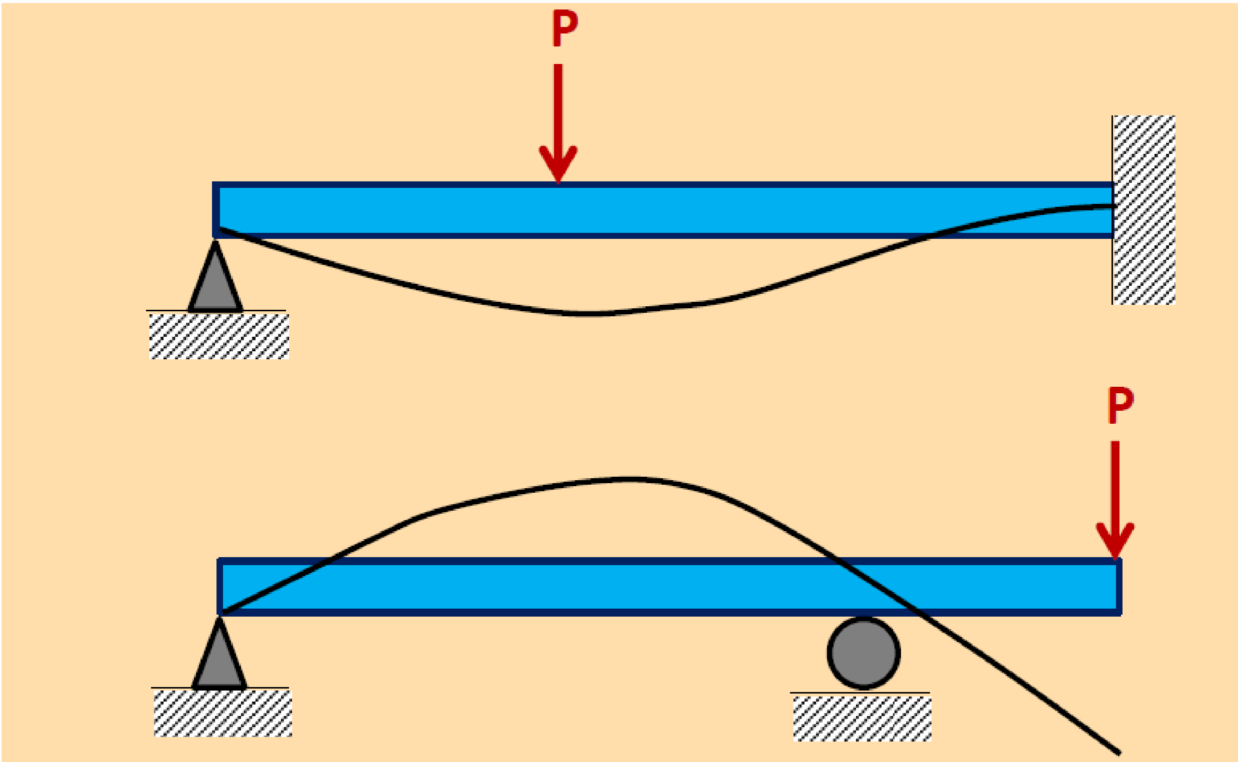


If the deflection value is too large, the beam will bend and then fail. Therefore it is vital that deflection must be limited within the allowable values as stipulated in the Standards

•The theory and background of deflection comes from **'<u>curvature</u>'**

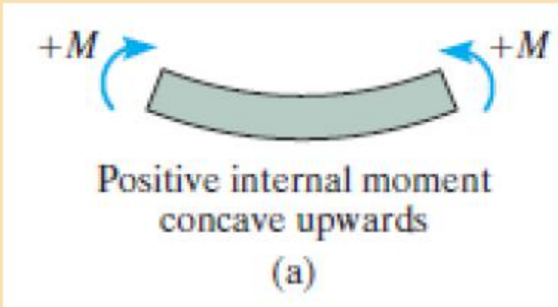
CURVATURE

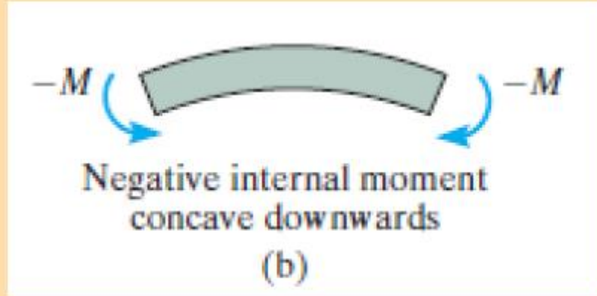
The deflection diagram of the longitudinal axis that passes through the centroid of each cross sectional area of the beam is called curvature or elastic curve, which is characterized by the deflection and slope along the curve

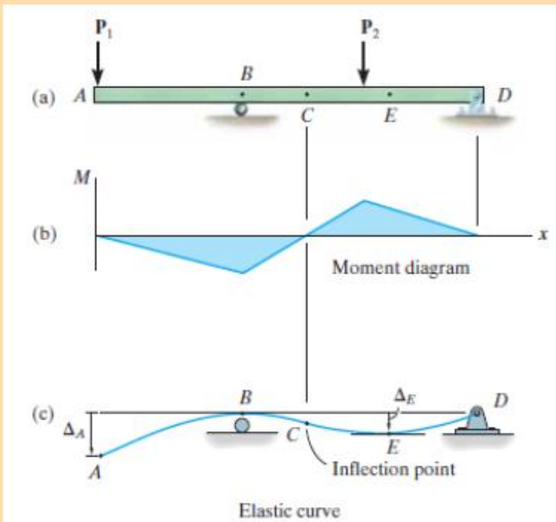


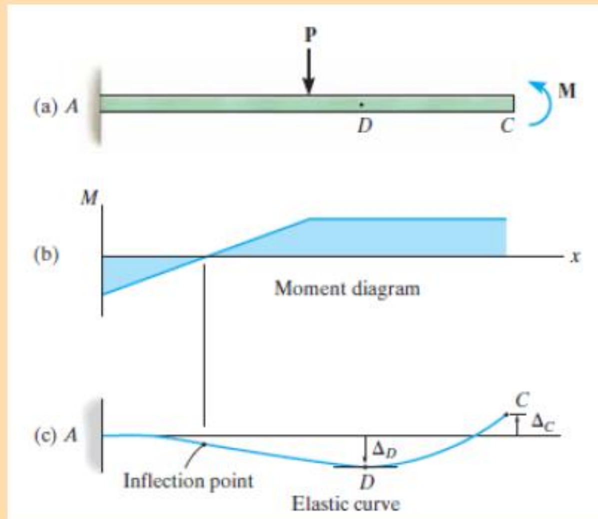
ELASTIC CURVATURE

- Moment-curvature relationship:
  - ➢ Sign convention:









CURVATURE

From the figure , if DE = L ; AB = DE = L

$$L' = A'B' = R\theta - (R-y)\theta$$

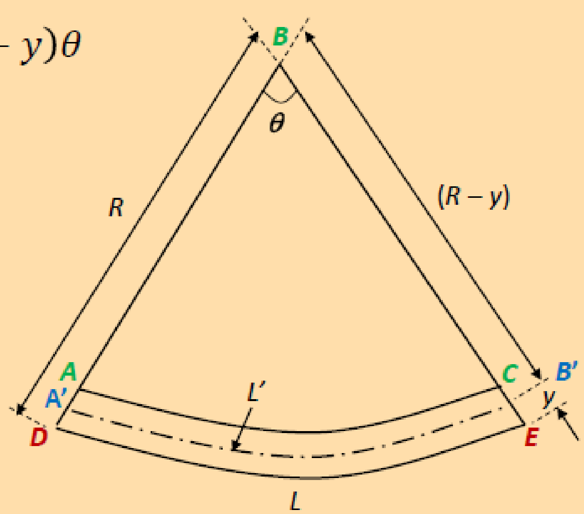
Displacement, $\delta = L - L' = R\theta - (R-y)\theta$

$\therefore\ \delta = y\theta$

From strain, $\varepsilon = \frac{\delta}{L} = \frac{y\theta}{R\theta} = \frac{y}{R}$

Therefore, curvature: $\frac{1}{R} = \frac{\varepsilon}{y}$

In elastic region; $E = \frac{\sigma}{\varepsilon}$ ➔ $\varepsilon = \frac{\sigma}{E}$



It is known that $\sigma = \frac{My}{I}$

Therefore : $\frac{1}{R} = \frac{\varepsilon}{y} = \frac{\sigma}{Ey} = \frac{My}{EIy}$

$$\boxed{\therefore \frac{1}{R} = \frac{M}{EI}}$$

where:

EI = Stiffness or Flexure Rigidity

(The higher the EI value, the stiffer the material → the smaller the curvature)

Several methods to compute deflections in beam

- Double integration method (*without* the use of singularity functions)
- Macaulay's Method (*with* the use of singularity functions)
- Moment area method
- Method of superposition
- Conjugate beam method
- Castigliano's theorem
- Work/Energy methods

Assumptions in Simple Bending Theory

- Beams are initially straight
- The material is homogenous and isotropic i.e. it has a uniform composition and its mechanical properties are the same in all directions
- The stress-strain relationship is linear and elastic
- Young's Modulus is the same in tension as in compression
- Sections are symmetrical about the plane of bending
- Sections which are plane before bending remain plane after bending

Non-Uniform Bending

- In the case of non-uniform bending of a beam, where bending moment varies from section to section, there will be shear force at each cross section which will induce shearing stresses
- Also these shearing stresses cause warping (or out-of plane distortion) of the cross section so that plane cross sections do not remain plane even after bending.

SLOPE & DEFLECTION BY DOUBLE INTEGRATION

- Deflection is influenced by *I*, *E* and *L* (and load)
- From $\frac{1}{R} = \frac{M}{EI}$ .... **(1)**
- Kinematic relationship between radius of curvature R and location x:

$$\frac{1}{R} = \frac{d^2y/dx^2}{\left[1 + \left({}^{dy}/_{dx}\right)^2\right]^{3/2}}$$

- But, in the case of elastic curve, the slope (*dy/dx*) is too small (≈ 0) and can be ignored. Then:

$$\frac{1}{R} = \frac{d^2y}{dx^2} \qquad \text{............. (2)}$$

- Substitute **Eq. (1)** into **Eq. (2)**:

$\frac{M}{EI} = \frac{d^2y}{dx^2}$ or $EI\frac{d^2y}{dx^2} = M$

Elastic curve differential equation (Moment Equation)

- After integration:

$$EI\frac{dy}{dx} = EI\theta = \int_x Mdx + C_1 \qquad \text{(Slope Equation)}$$

- Double Integration will produce:

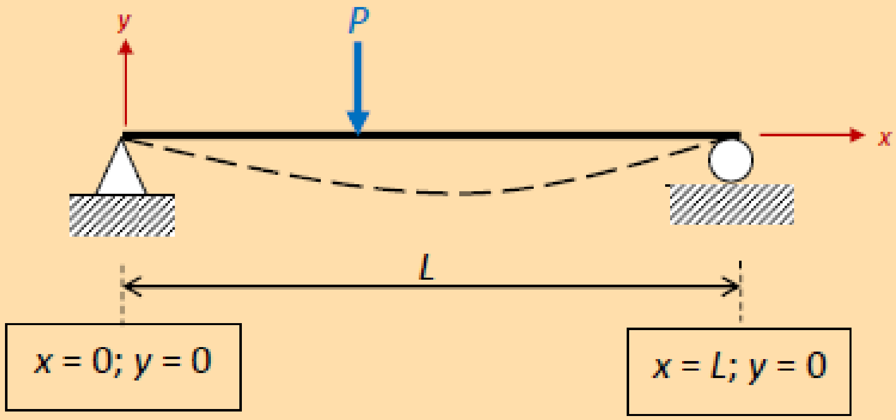
$$EIy = \int_x [_x Mdx]dx + C_1x + C_2 \qquad \text{(Deflection Equation)}$$

where C_1 and C_2 are the constant to be determined from the **boundary conditions**

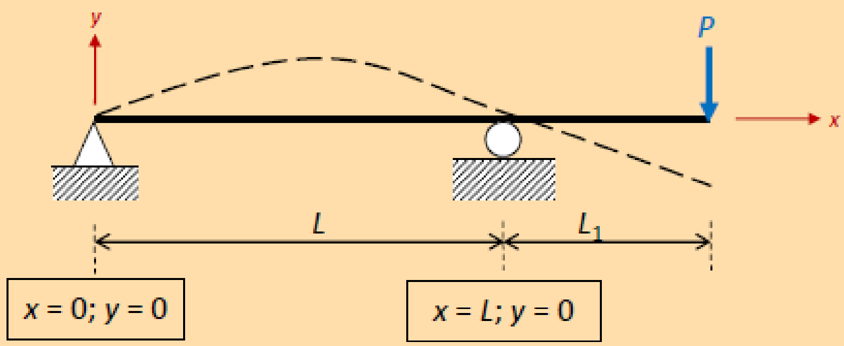
BOUNDARY CONDITIONS

- The integration constants can be determined by imposing the boundary conditions, or continuity condition at specific locations
- 3 beam cases are considered:

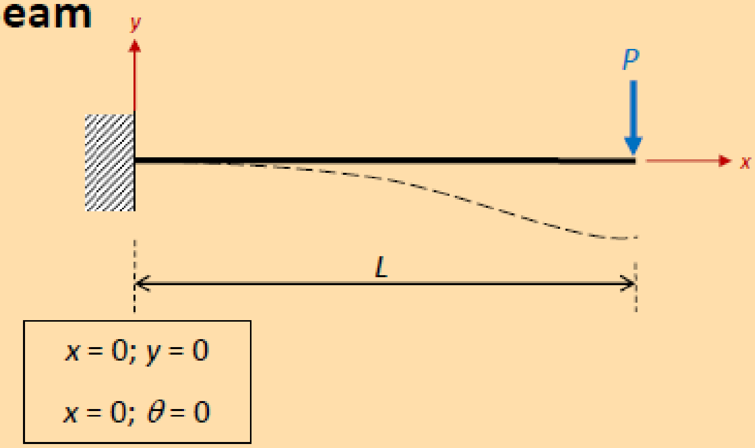
I. Simply Supported Beam



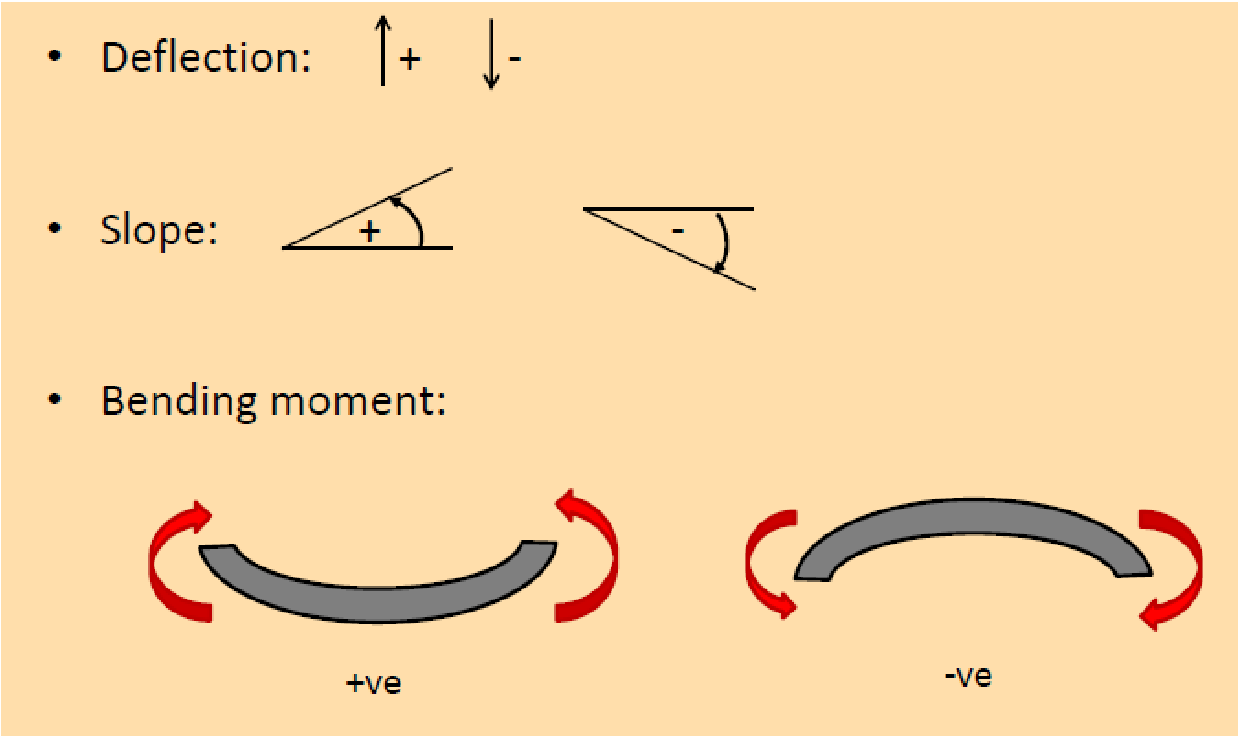
II. Overhanged Beam



III. Cantilever Beam

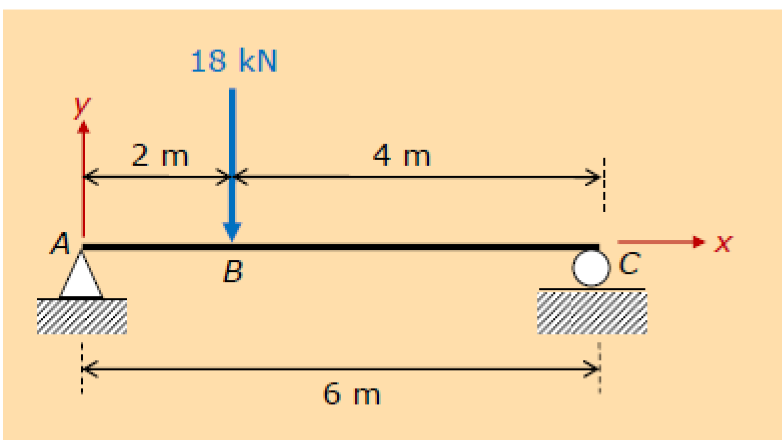


SIGN CONVENTIONS



Example: 01 (Double Integration)

Determine the mid-span deflection of beam shown below. Given E = 20 kN/mm2 and I = 1600 x 106 mm^4.



SOLUTION

Determine the Reaction Forces at A and C

Taking moment at C; $\Sigma M_C = 0$

$V_A\ (6) - 18(4) = 0$

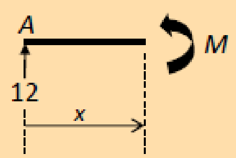
$V_A = 12$ kN

$\Sigma F_y = 0$

$V_A + V_C = 0$

$V_C = 6$ kN

Segment AB ($0 \leq x \leq 2$)



$$M = 12x$$

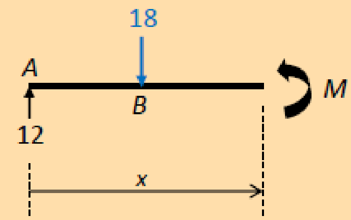
$$EI\frac{d^2y}{dx^2} = M = 12x$$

$$EI\frac{dy}{dx} = \frac{12x^2}{2} + C_1$$

$$EIy = \frac{12x^3}{6} + C_1x + C_2$$

$$EIy_{AB} = 2x^3 + C_1x + C_2 \quad \text{....(1)}$$

Segment BC ($2 \leq x \leq 6$)



$$M = 12x - 18(x-2) = -6x + 36$$

$$EI\frac{d^2y}{dx^2} = M = -6x + 36$$

$$EI\frac{dy}{dx} = \frac{-6x^2}{2} + 36x + C_3$$

$$EIy = \frac{-6x^3}{6} + \frac{36x^2}{2} + C_3x + C_4$$

$$EIy_{BC} = -x^3 + 18x^2 + C_3x + C_4 \quad \text{....(2)}$$

Boundary Conditions

When $x = 0$, $y = 0$ .... **(A)**

When $x = 6$, $y = 0$ .... **(B)**

Matching Conditions

At $x = 2$; $\left(\frac{dy}{dx}\right)_{AB} = \left(\frac{dy}{dx}\right)_{BC}$ and $y_{AB} = y_{BC}$

Substitute **(A)** into **Eq. (1)**: $EI(0) = 2(0) + C_1(0) + C_2$ ➔ $C_2 = 0$

Substitute **(B)** into **Eq. (2)**: $EI(0) = -(6)^3 + 18(6)^3 + 6C_3 + C_4$

$C_4 = -432 - 6C_3$ **....(3)**

From the Matching Conditions:

$$\frac{12(2)^2}{2} + C_1 = \frac{-6(2)^2}{2} + 36(2) + C_3$$

$C_1 = 36 + C_3$ **....(4)**

$$2(2)^3 + C_1(2) = -(2)^3 + 18(2)^2 + C_3(2) + C_4$$

$2C_1 = 48 + 2C_3 + C_4$ **....(5)**

Substitute **Eq. (3)** and **Eq. (4)** into **Eq. (5)**:

$2(36 + C_3) = 48 + 2C_3 + (-432 - 6C_3)$ ➔ $C_3 = -76$

From **Eq. (3)**: C4 = -432 – 6(-76) = 24

From **Eq. (4)**: C1 = 36 – 76 = -40

Therefore:

$$y = \frac{1}{EI}[2x^3 - 40x] \qquad (0 \le x \le 2)$$

$$y = \frac{1}{EI}[-x^3 + 18x^2 - 76x + 24] \qquad (2 \le x \le 6)$$

To determine the deflection at mid-span, x = 3 m:

$$y_{3m} = \frac{1}{EI}[-(3)^3 + 18(3)^2 - 76(3) + 24] = \frac{-69}{EI}$$

$$= \frac{-69}{(20\times10^6)(1600\times10^{-6})} = -0.00215 \text{ m} = -\mathbf{2.15 \text{ mm}} \qquad \textbf{(ANS)}$$

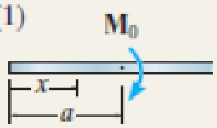
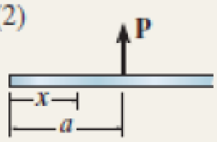
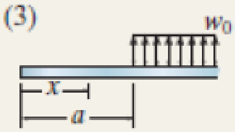
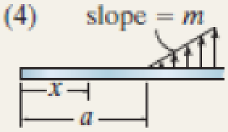
**Negative deflection value shows downward direction*

Conclusion:

Each different load produces different section and 2 constant unknowns. Say if we have 4 sections (8 unknowns).Therefore, this method is not practical.

MAC CAULAY METHOD

This is a simplified method based on the double integration concept. In this method, only **ONE** section will be considered which is at the last loading type.

Loading	Loading Function $w = w(x)$	Shear $V = \int w(x)dx$	Moment $M = \int V dx$
(1) 	$w = M_0\langle x-a\rangle^{-2}$	$V = M_0\langle x-a\rangle^{-1}$	$M = M_0\langle x-a\rangle^{0}$
(2) 	$w = P\langle x-a\rangle^{-1}$	$V = P\langle x-a\rangle^{0}$	$M = P\langle x-a\rangle^{1}$
(3) 	$w = w_0\langle x-a\rangle^{0}$	$V = w_0\langle x-a\rangle^{1}$	$M = \frac{w_0}{2}\langle x-a\rangle^{2}$
(4) 	$w = m\langle x-a\rangle^{1}$	$V = \frac{m}{2}\langle x-a\rangle^{2}$	$M = \frac{m}{6}\langle x-a\rangle^{3}$

USE OF CONTINUOUS FUNCTIONS

- *Mac Caulay* functions

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases}$$

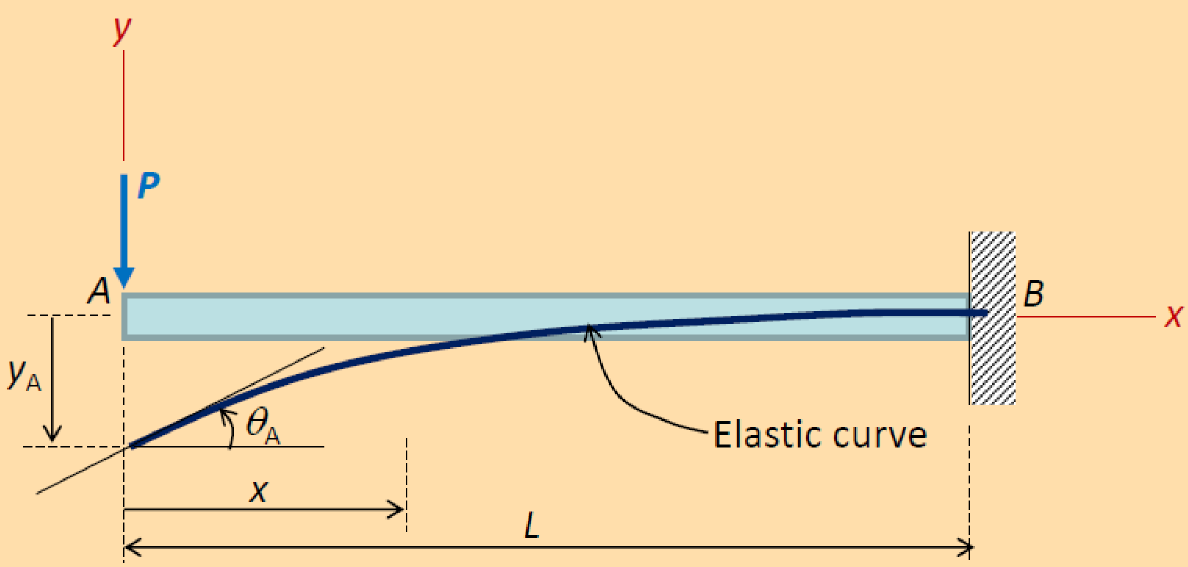
$$n \geq a$$

- Integration of *Mac Caulay* functions:

$$\int \langle x-a\rangle^n \, dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C$$

<u>Example 02:</u> mac caulay method

The cantilever beam shown in the figure below is subjected to a vertical load ***P*** at its end. Determine the equation of the elastic curve. *EI* is constant.



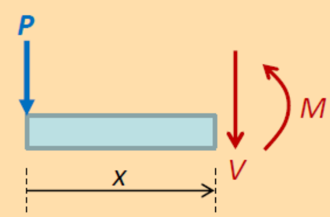
- From the free-body diagram, with ***M*** acting in the *positive direction* as shown in figure, we have M = -Px

- Integrating twice yields;

$$EI\frac{d^2y}{dx^2} = -Px \quad \text{.... (1)}$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1 \quad \text{.... (2)}$$

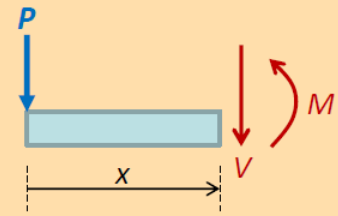
$$EIy = -\frac{Px^3}{6} + C_1x + C_2 \quad \text{.... (3)}$$



- Using the boundary conditions $dy/dx = 0$ at $x = L$ and $y = 0$ at $x = L$, **Eq. (2)** and **Eq. (3)** becomes;

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1 L + C_2$$



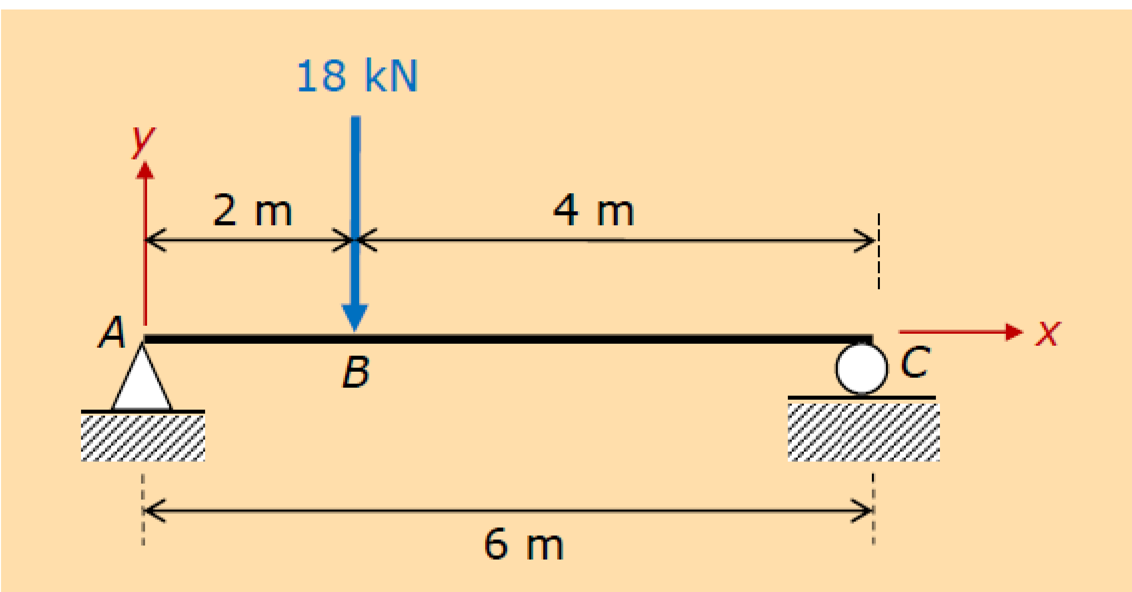
Therefore, $C_1 = \frac{PL^2}{2}$ and $C_2 = -\frac{PL^3}{3}$

- Substituting these results, we get;

$$\boldsymbol{\theta = \frac{P}{2EI}(L^2 - x^2)} \textbf{ and } \boldsymbol{y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)} \quad \textbf{(ANS)}$$

Example: 03 (Mac Caulay Method)

Determine the mid-span deflection of beam shown below. Given E = 20 kN/mm2 and
I = 1600 x 106 mm^4.



Determine the Reaction Forces at A and C

Taking moment at C; $\Sigma M_C = 0$

$V_A\ (6) - 18(4) = 0$

$\therefore\ V_A = 12$ kN

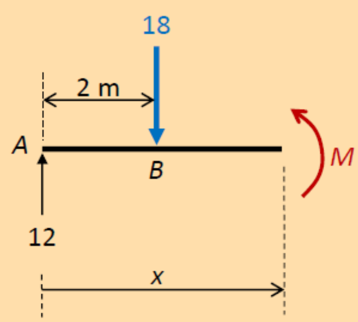
$\Sigma F_y = 0$

$V_A + V_C\ = 0$

$\therefore\ V_C = 6$ kN

Consider to make a section after last load, i.e. in region BC (section made from left to right). Therefore, the moment equation:

$$M_x = 12x - 18(x - 2)$$



Moment: $M = 12x - 18\langle x - 2\rangle$

$$EI\frac{d^2y}{dx^2} = 12x - 18\langle x - 2\rangle \qquad \text{.... (1)}$$

Slope: $EI\frac{dy}{dx} = \frac{12x^2}{2} - \frac{18\langle x-2\rangle^2}{2} + C_1$

$$= 6x^2 - 9\langle x - 2\rangle^2 + C_1 \qquad \text{.... (2)}$$

Deflection: $EIy = \frac{12x^3}{6} - \frac{18\langle x-2\rangle^3}{6} + C_1\,x + C_2$

$$= 2x^3 - 3\langle x - 2\rangle^3 + +C_1x + C_2 \qquad \text{.... (3)}$$

Boundary Conditions

At *x* = 0, *y* = 0 and from Eq. (3):

$EI(0) = 2(0)^3 - 3\langle 0 - 2\rangle^3 + +C_1(0) + C_2$

$\therefore$ $C_2 = 0$

At *x* = 6, *y* = 0 and from Eq. (3):

$EI(6) = 2(6)^3 - 3\langle 6 - 2\rangle^3 + +C_1(6) + 0$

$\therefore$ $C_1 = -40$

Therefore, **Eq. (2)** becomes:

$$EI\frac{dy}{dx} = 6x^2 - 9\langle x-2\rangle^2 - 40 \qquad \text{.... (4)}$$

and **Eq. (3)** becomes:

$$EIy = 2x^3 - 3\langle x-2\rangle^3 - 40x \qquad \text{.... (5)}$$

At mid-span, *x* = 3 m:

$$EIy_{3m} = 2(3)^3 - 3\langle 3-2\rangle^3 - 40(3) = -69$$

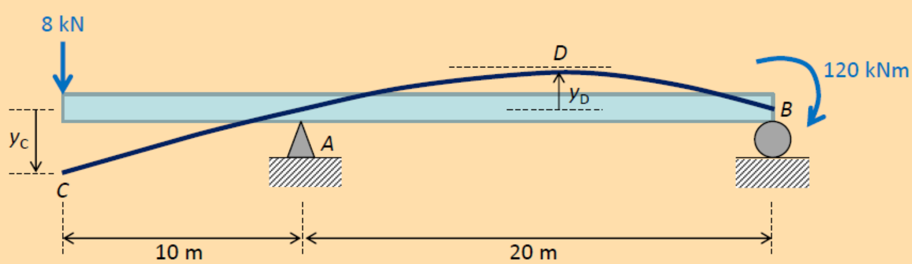
$$\therefore\ y_{3m} = \frac{-69}{(20\times10^6)(1600\times10^{-6})} = -0.00215 \text{ m} = \mathbf{-2.15\ m} \qquad \textbf{(ANS)}$$

$$EI\left(\frac{dy}{dx}\right)_{3m} = 6(3)^2 - 9\langle 3-2\rangle^2 - 40 = 5$$

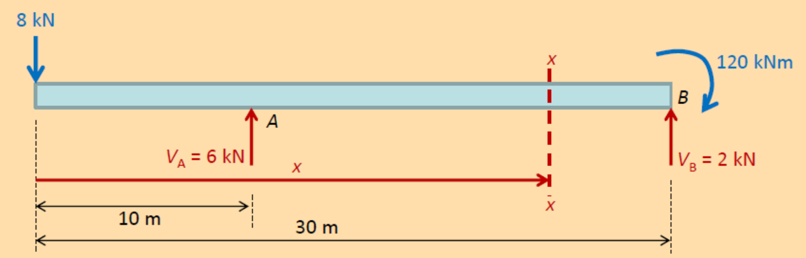
$$\therefore\ \left(\frac{dy}{dx}\right)_{3m} = \frac{5}{(20\times10^6)(1600\times10^{-6})} = \mathbf{1.56\times10^{-4}\ rad} \qquad \textbf{(ANS)}$$

Example:04

Determine the maximum deflection of the beam shown in the figure below. *EI* is constant



The beam deflects as shown in the figure. The boundary conditions require zero displacement at *A* and *B* (y_A = y_B = 0).



- The moment equation section at *x*-*x* is:

$$M = -8x + 6\langle x-10\rangle$$
$$= (-8x + 6\langle x-10\rangle) \text{ kNm}$$

Integrating twice yields:

$$EI\frac{d^2y}{dx^2} = -8\langle x\rangle^1 + 6(x-10)^1 \qquad \text{.... (1)}$$

$$EI\frac{dy}{dx} = -4x^2 + 3\langle x-10\rangle^2 + C_1 \qquad \text{.... (2)}$$

$$EIy = -\frac{4}{3}x^3 + \langle x-10\rangle^3 + C_1x + C_2 \qquad \text{.... (3)}$$

Boundary Conditions

y = 0 at x = 10 m and from **Eq. (3)**:

$$0 = -1333 + \langle 10-10\rangle^3 + C_1(10) + C_2$$

y = 0 at x = 30 m and from **Eq. (3)**:

$$0 = -36000 + \langle 30-10\rangle^3 + C_1(30) + C_2$$

$\therefore$ C_1 = 1333 and C_2 = –12000

From **Eq. (2):**

$$EI\frac{dy}{dx} = -4x^2 + 3\langle x-10\rangle^2 + 1333 \qquad \text{.... (4)}$$

From **Eq. (3)**:

$$EIy = -\frac{4}{3}x^3 + \langle x-10\rangle^3 + 1333x - 12000 \qquad \text{.... (5)}$$

To obtain the deflection at C, x = 0. Therefore, from **Eq. (5)**:

$$\therefore\ \boldsymbol{y_C = -\frac{12000}{EI}}\ \textbf{kNm}^3 \qquad \textbf{(ANS)}$$

**The negative sign indicates that deflection is downward*

To determine the length at point *D*, use **Eq. (4)** with x > 10 and $\left(\frac{dy}{dx}\right) = 0$

$$0 = -4{x_D}^2 + 3\langle x_D - 10\rangle^2 + 1333$$
$$4{x_D}^2 + 60x_D - 1633 = 0$$

Solving for the positive root, x_D = 20.3 m

Hence from **Eq. (5)**:

$$EIy_D = -\frac{4}{3}(20.3)^3 + \langle 20.3 - 10\rangle^3 + 1333(20.3) - 12000$$

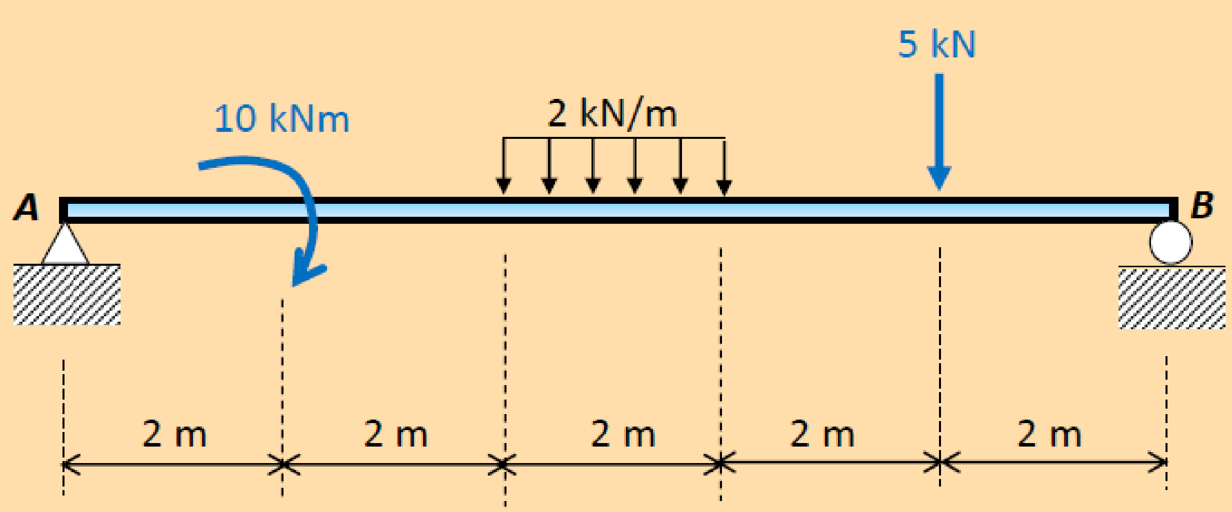
$\boldsymbol{y_D = \frac{5006}{EI}}$ **kNm³** **(ANS)**

**The positive sign indicates that deflection is upward*

Comparing y_D with y_C, $\boldsymbol{y_{max} = y_C}$ **(ANS)**

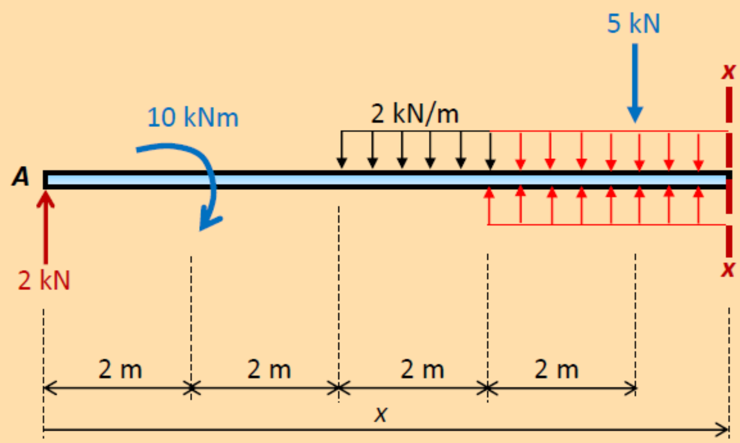
Example:5

Determine the slope and deflection at x = 3 m. Also, determine the location and magnitude of the maximum deflection. Given EI = 4000 kNm^2.



Cut at section *x-x* as shown in the figure. Therefore, the moment equation is given as:

$$M_{x-x} = 2\langle x\rangle^1 + 10\langle x-2\rangle^0 - \frac{2\langle x-4\rangle^2}{2} + \frac{2\langle x-6\rangle^2}{2} - 5\langle x-8\rangle^1$$



Determine the reaction forces at A and B

Taking moment at A, $\Sigma M_A = 0$

$10 + (2)(2)(5) + (5)(8) - V_B(10) = 0$

$V_B = 7$ kN

$\Sigma F_y = 0$

$V_A - 2(2) - 5 + V_B = 0$

$V_A = 2$ kN

$$EI\frac{d^2y}{dx^2} = M = 2\langle x\rangle^1 + 10\langle x-2\rangle^0 - \frac{2\langle x-4\rangle^2}{2} + \frac{2\langle x-6\rangle^2}{2} - 5\langle x-8\rangle^1$$
$$EI\frac{dy}{dx} = \theta = \frac{2\langle x\rangle^2}{2} + \frac{10\langle x-2\rangle^1}{1} - \frac{2\langle x-4\rangle^3}{6} + \frac{2\langle x-6\rangle^3}{6} - \frac{5\langle x-8\rangle^2}{2} + C_1$$
$$EIy = \frac{2\langle x\rangle^3}{6} + \frac{10\langle x-2\rangle^2}{2} - \frac{2\langle x-4\rangle^4}{24} + \frac{2\langle x-6\rangle^4}{24} - \frac{5\langle x-8\rangle^3}{6} + C_1x + C_2$$

Boundary Conditions

At x = 0 m, y = 0:

$\therefore$ C_2 = 0

At x = 10 m, y = 0:

$$0 = \frac{2\langle 10\rangle^3}{6} + \frac{10\langle 8\rangle^2}{2} - \frac{2\langle 6\rangle^4}{24} + \frac{2\langle 4\rangle^4}{24} - \frac{5\langle 2\rangle^3}{6} + 10C_1$$

$\therefore$ C_1 = –56

Slope Equation:

$$EI\theta = \langle x\rangle^2 + 10\langle x-2\rangle^1 - \frac{\langle x-4\rangle^3}{3} + \frac{\langle x-6\rangle^3}{3} - \frac{5\langle x-8\rangle^2}{2} - 56$$

Deflection Equation:

$$EIy = \frac{\langle x\rangle^3}{3} + 5\langle x-2\rangle^2 - \frac{\langle x-4\rangle^4}{12} + \frac{\langle x-6\rangle^4}{12} - \frac{5\langle x-8\rangle^3}{6} - 56x$$

Now, we can find the slope and deflection at x = 3 m. Given EI = 4000 kNm^2

$\theta = \frac{1}{4000}[\langle 3\rangle^2 + 10\langle 3-2\rangle^1 - 56]$= –0.00925 rad

$y = \frac{1}{4000}\left[= \frac{\langle 3\rangle^3}{3} + 5\langle 3-2\rangle^2 - 56(3)\right]$= –0.00385 m

To determine the position and magnitude of the maximum deflection, y_{max} when $\left(\frac{dy}{dx}\right) = 0$:

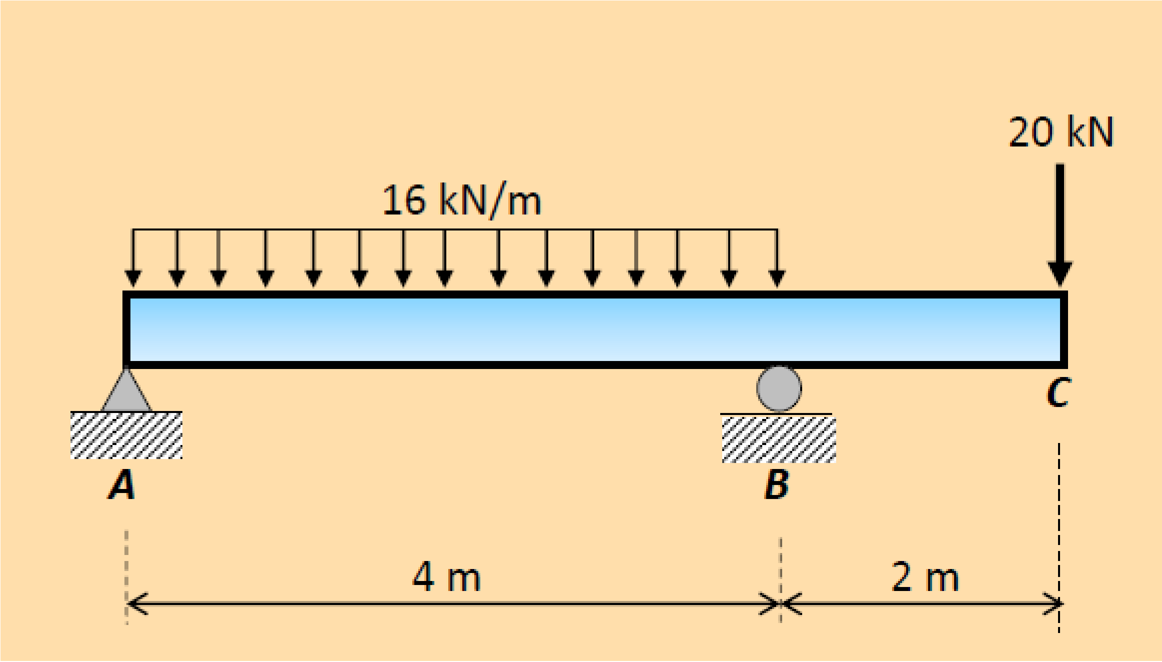
x (m)	$EI\left(\frac{dy}{dx}\right)$
5	-1.3
5.5	+8.13
5.2	+2.5

From interpolation; x_{max} = 5.1 m

Therefore, at x_{max} = 5.1 m; $\boldsymbol{y_{max}}$ **= –48.3 mm** **(ANS)**

Example:06

Find the deflection equation for the given beam. Then, determine the maximum deflection at mid-span along span *AB*. Given E = 200 kN/mm^2 and I = 10 x 10^6 mm^4.

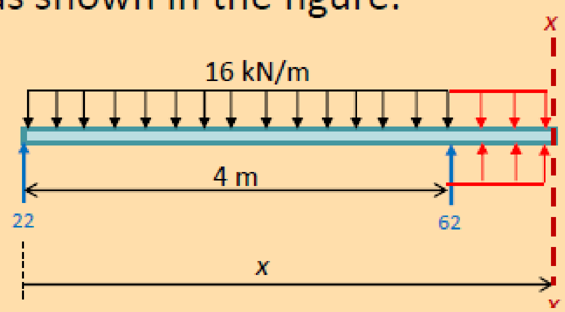


Determine the reactions forces at A and B

Taking moment at B, $\Sigma M_B = 0$; $20(2) - (16)(4)(2) + V_A(4) = 0$
$\therefore V_A = 22$ kN

$\Sigma F_y = 0$; $V_A + V_B - 16(4) - 20 = 0$
$\therefore V_B = 62$ kN

Cut at section *x-x* as shown in the figure.



Therefore, the moment equation is given as:

$$M_{x-x} = 22\langle x\rangle^1 + 62\langle x-4\rangle^1 - 16x\left\langle\frac{x}{2}\right\rangle + 16\langle x-4\rangle\frac{\langle x-4\rangle}{2}$$

$$EI\frac{d^2y}{dx^2} = M = 22\langle x\rangle^1 + 62\langle x-4\rangle^1 - 8\langle x\rangle^2 + 8\langle x-4\rangle^2$$

$$EI\frac{dy}{dx} = EI\theta = \frac{22\langle x\rangle^2}{2} + \frac{62\langle x-4\rangle^2}{2} - \frac{8\langle x\rangle^3}{3} + \frac{8\langle x-4\rangle^3}{3} + C_1$$

$$EIy = \frac{22\langle x\rangle^3}{6} + \frac{62\langle x-4\rangle^3}{6} - \frac{8\langle x\rangle^4}{12} + \frac{8\langle x-4\rangle^4}{12} + C_1x + C_2$$

$$= \frac{11\langle x\rangle^3}{3} + \frac{31\langle x-4\rangle^3}{3} - \frac{2\langle x\rangle^4}{3} + \frac{2\langle x-4\rangle^4}{3} + C_1x + C_2$$

Boundary Condition at A

At $x = 0$ m, $y = 0$:

$\therefore C_2 = 0$

Boundary condition at B

At $x = 4$ m, $y = 0$:

$$EI(0) = \frac{11\langle 4\rangle^3}{3} + \frac{31\langle 4-4\rangle^3}{3} - \frac{2\langle 4\rangle^4}{3} + \frac{2\langle 4-4\rangle^4}{3} + 4C_1$$

$\therefore C_1 = -16$

Therefore, the slope and deflection equations are:

$$EI\frac{dy}{dx} = 11\langle x\rangle^2 + 31\langle x-4\rangle^2 - \frac{8\langle x\rangle^3}{3} + \frac{8\langle x-4\rangle^3}{3} - 16$$

$$EIy = \frac{11\langle x\rangle^3}{3} + \frac{31\langle x-4\rangle^3}{3} - \frac{2\langle x\rangle^4}{3} + \frac{2\langle x-4\rangle^4}{3} - 16x$$

To determine the maximum deflection and where it occurred, $\left(\frac{dy}{dx}\right) = 0$. Therefore:

$$EI(0) = 11\langle x\rangle^2 + 31\langle x-4\rangle^2 - \frac{8\langle x\rangle^3}{3} + \frac{8\langle x-4\rangle^3}{3} - 16$$

Consider the maximum deflection occurs along span AB ($0 \leq x \leq 4$)

$$EI(0) = 11\langle x\rangle^2 + \cancel{31\langle 0\rangle^2} - \frac{8\langle x\rangle^3}{3} + \cancel{\frac{8\langle 0\rangle^3}{3}} - 16$$

$0 = 11x^2 - \frac{8x^3}{3} - 16$ or

$8x^3 - 33x^2 + 48 = 0$

By try and error: $x_{max} = 1.52$ m from A.

Therefore, the maximum deflection occurs when x = 1.52 m from A.
To calculate the maximum deflection:

$$EIy_{max} = \frac{11\langle 1.52\rangle^3}{3} + \cancel{\frac{31\langle 1.52-4\rangle^3}{3}} - \frac{2\langle 1.52\rangle^4}{3} + \cancel{\frac{2\langle 1.52-4\rangle^4}{3}} - 16(1.52)$$

$$EIy_{max} = -15$$

$$y_{max} = \frac{-15}{(200\times 10^6)(10\times 10^{-6})} = -0.0075 \text{ m} = \mathbf{-7.5 \text{ mm (downward)}}$$

MOMENT AREA METHOD

- Based on the properties of elastic curve and bending moment diagram.
- Suitable to use for determining deflection and slope at a **particular point**.
- Also suitable for beam with **different cross-section**.
- There are **two important theorems** used in this method

Theorem 1

The angle between the tangents at any two points on the elastic curve equals the area under the $\frac{\mathbf{M}}{\mathbf{EI}}$ diagram between these two points.

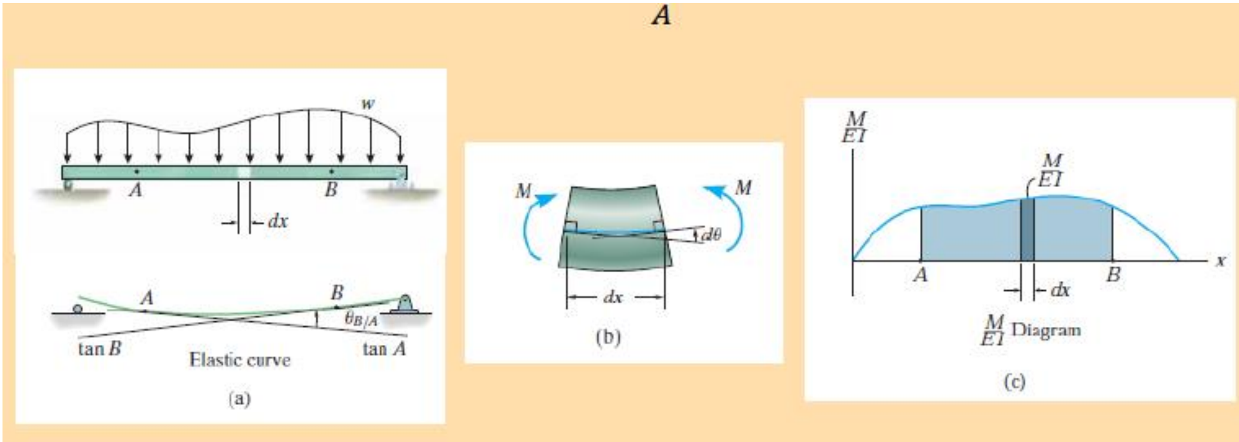
$$\mathrm{EI}\frac{d^2y}{dx^2} = EI\frac{d}{dx}\left(\frac{dy}{dx}\right) = M$$

$$d\theta = \left(\frac{M}{EI}\right)dx$$

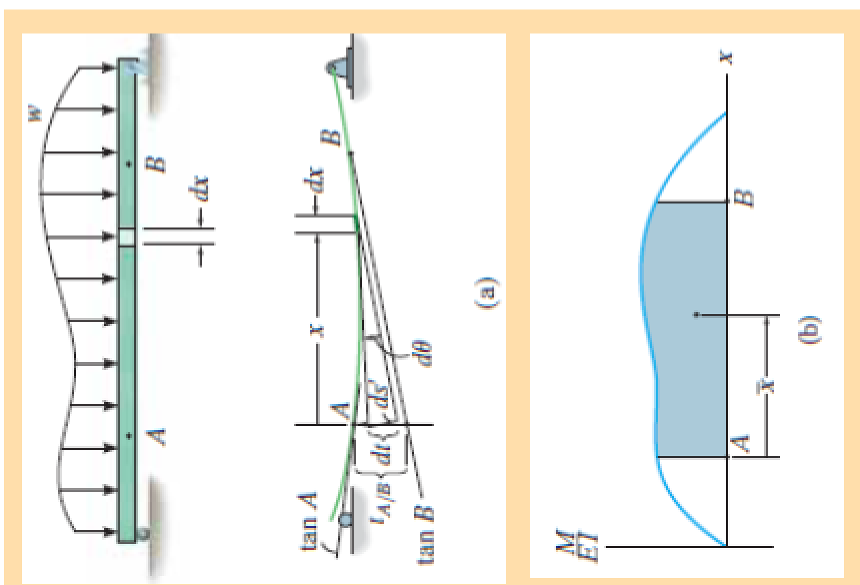
- Since $\theta \approx \frac{dy}{dx}$, so $d\theta = \left(\frac{M}{EI}\right)dx$
- Therefore, $\theta_{AB} = \int_{x_A}^{x_B} \frac{M}{EI}dx = \frac{1}{EI}\cdot M_{A-B}\ Area$

This equation forms the basis for the **first moment-area theorem**:

$$\theta_{AB} = \theta_{B/A} = \int_A^B \frac{M}{EI}dx$$

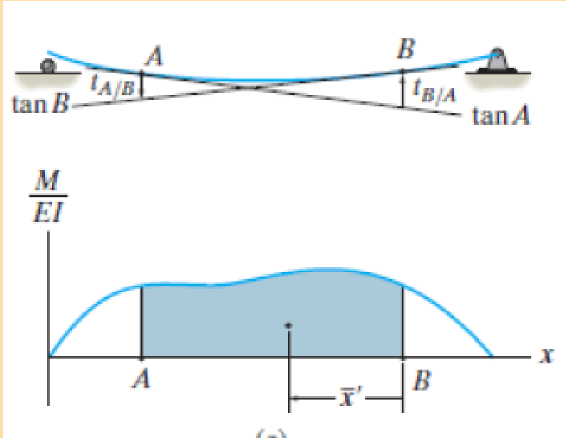


Theorem 2
The vertical deviation of the tangent at point ***A*** on the elastic curve with respect to the tangent extended from another point ***B*** equals the moment of the area under the $\frac{\mathbf{M}}{\mathbf{EI}}$ diagram between these two points (***A*** and ***B***). This moment is computed about point, ***A*** where the vertical deviation $t_{A/B}$ is to be determined.



- The vertical deviation of the tangent at ***A*** with respect to the tangent at ***B*** is given as:

$$t_{A/B} = \int_{A}^{B} x \frac{M}{EI} dx$$



(c)

- Then:

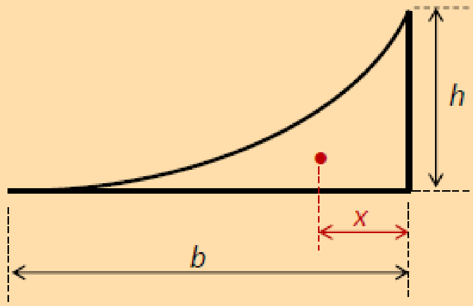
$$t_{A/B} = \bar{x} \int_{A}^{B} \frac{M}{EI} dx$$

where x is the location of the centroid of the shaded area $\int \left(\frac{M}{EI}\right) dx$ between ***A*** and ***B***.

- Centroid and Area

$$Area, A = \frac{bh}{(n+1)}$$

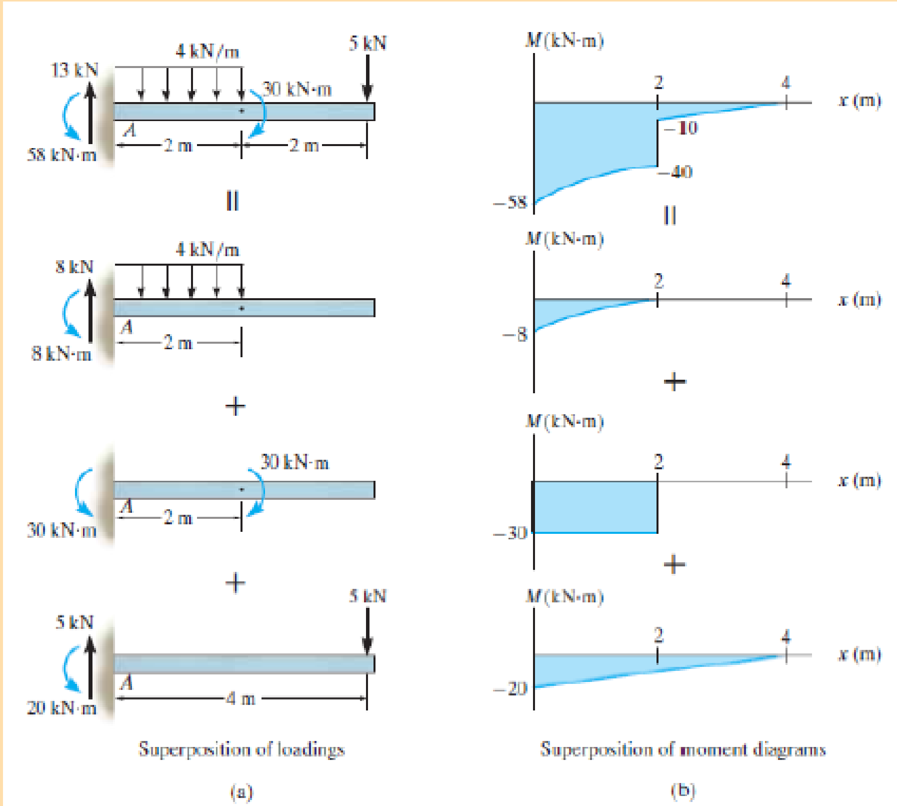
$$Centroid, \bar{x} = \frac{b}{(n+2)}$$



***Notes:**

- *Draw BMD separately for each load, with one reference point*
- *Write the bending moment equation in f(x) terms*
- *Upward force produces positive bending moment and vice versa*

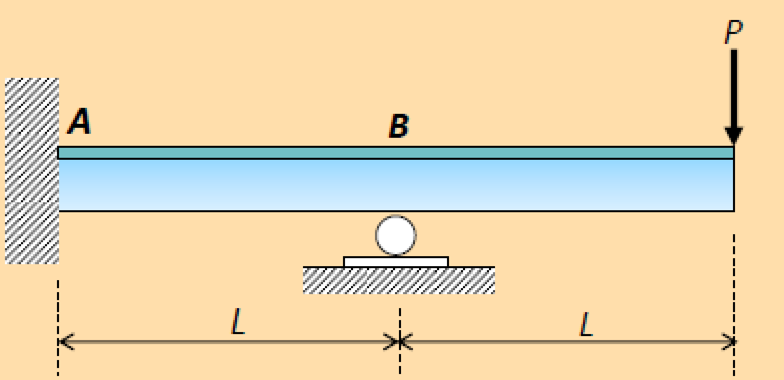
Procedures:



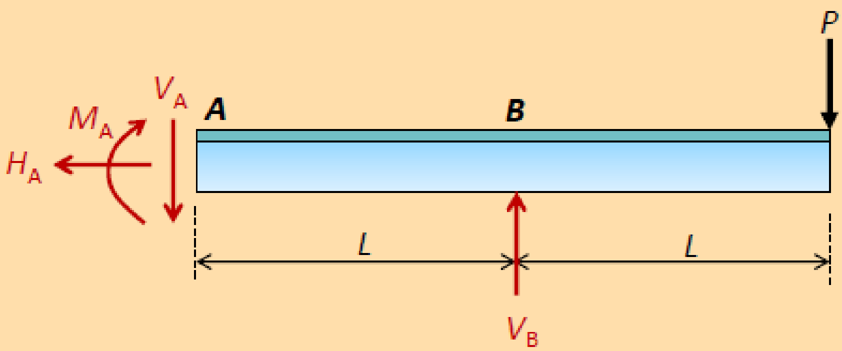
Superposition of loadings (a) Superposition of moment diagrams (b)

Example:07

The beam is subjected to the concentrated force shown in the figure. Determine the reactions at the supports. EI is constant.

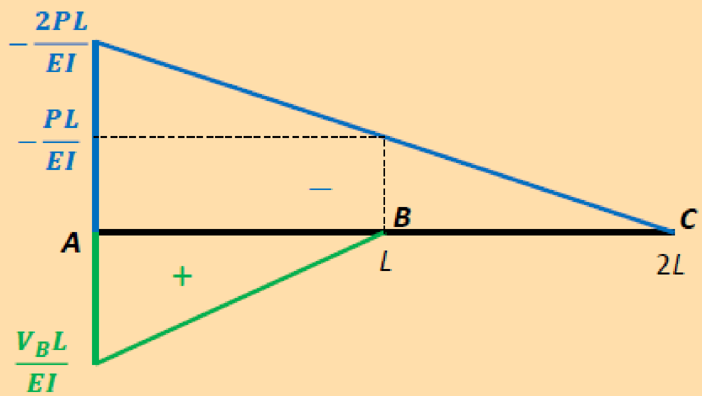


- The free-body diagram is shown in Fig. (b).



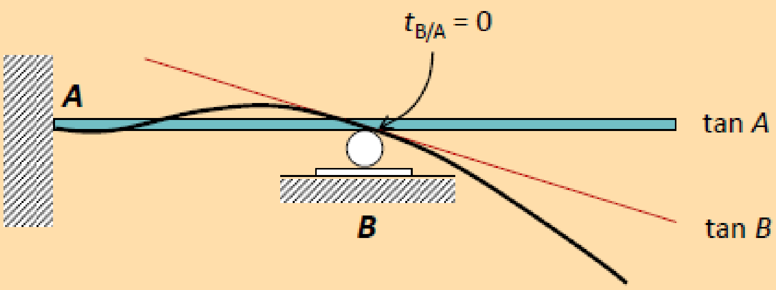
(b)

- Using the method of superposition, the separate $\left(\frac{M}{EI}\right)$ diagrams for the redundant reaction $\boldsymbol{V}_{\mathbf{B}}$ and the load $\boldsymbol{P}$ are shown in Fig. (c).



(c)

- The elastic curve for the beam is shown in Fig. (d).



(d)

- Applying **Theorem 2**, we have:

$$t_{B/A} = \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{V_B L}{EI}\right)L\right] + \left(\frac{L}{2}\right)\left[\frac{-PL}{EI}(L)\right] + \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{-PL}{EI}\right)(L)\right] = 0$$

V_B = 2.5P (ANS)

- Using this result, the reactions at ***A*** on the free-body diagram, of Fig. (b), are:

$+\rightarrow$	$\Sigma F_x = 0;$	**$H_A = 0$**	**(ANS)**
$+\uparrow$	$\Sigma F_y = 0;$	$-V_A + 2.5P - P = 0$	
		$V_A = 1.5P$	**(ANS)**
$+\circlearrowleft$	$\Sigma M_A = 0;$	$-M_A + 2.5P(L) - P(2L) = 0$	
		$M_A = 0.5PL$	**(ANS)**

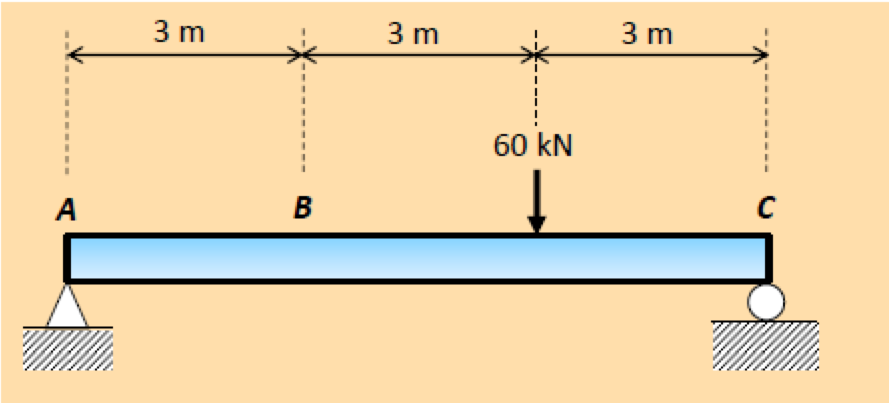
Example:08

The beam shown in the figure is pin supported at ***A*** and roller at ***B***. A point load of 60 kN is applied at 6 m from ***A***. Determine:

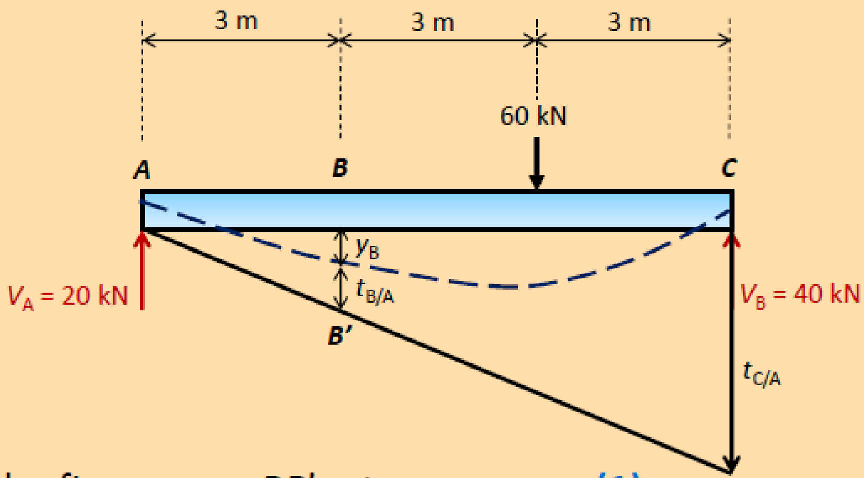
i)Deflection at ***B***

ii)Slope at ***B***

iii)Maximum deflection



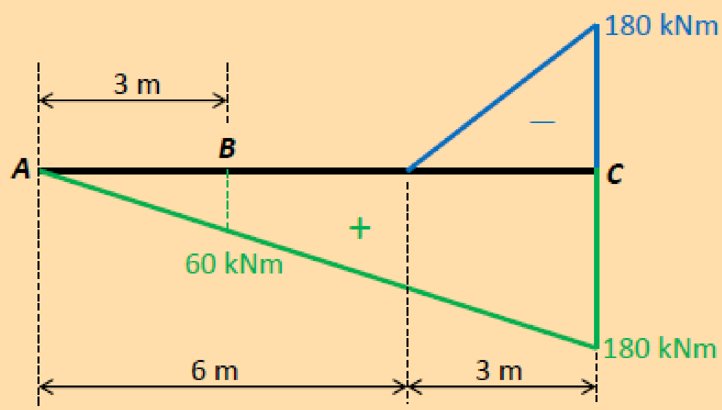
The deflection and slope diagram of the beam. Reference point is taken from ***A***.



From the figure, $y_B = BB' - t_{B/A}$ .... **(1)**

where $\frac{BB'}{3} = \frac{t_{C/A}}{9}$ ➔ $BB' = \frac{t_{C/A}}{3}$ .... **(2)**

The BMD for each loading i.e. V_A **= 20 kN** and ***P* = 60 kN**



$$t_{B/A} = +\frac{1}{EI}\left[\frac{1}{2}\times 3\times 60\left(\frac{1}{3}\times 3\right)\right] = +\frac{90}{EI}$$

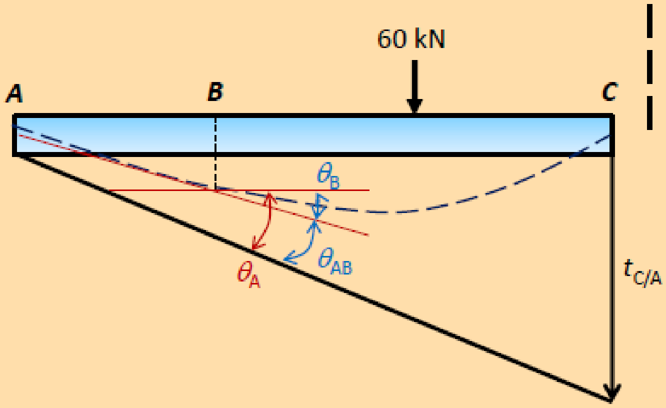
$$t_{C/A} = +\frac{1}{EI}\left[\frac{1}{2}\times 9\times 180\left(\frac{1}{3}\times 9\right) - \frac{1}{2}\times 3\times 180\left(\frac{1}{3}\times 3\right)\right] = +\frac{2160}{EI}$$

$$BB' = \frac{1}{3}\left(\frac{2160}{EI}\right) = \frac{720}{EI}$$

$\therefore$ From **Eq. (1)**; $y_B = \frac{720}{EI} - \frac{90}{EI} = \frac{630}{EI}(\downarrow)$

Let say EI = 30,000 kNm^2

$\therefore \boldsymbol{y_B = 21\ mm}\ (\downarrow)$ **(ANS)**



$\theta_A = \theta_{AB} + \theta_B$

$\therefore\ \theta_B = \theta_A - \theta_{AB}$

$$\theta_A \approx tan\theta_A = \frac{t_{C/A}}{9} = \frac{2160}{9EI}$$

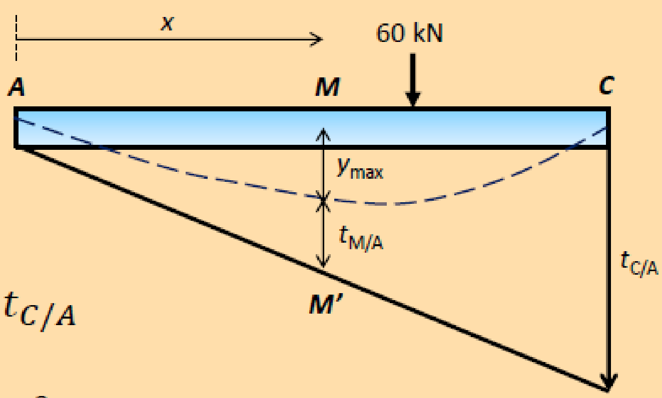
$$\theta_{AB} = +\frac{1}{EI}\left[\frac{1}{2} \times 3 \times 60\right] = +\frac{90}{EI}$$

$$\therefore\ \theta_B = \frac{2160}{9EI} - \frac{90}{EI} = \frac{150}{EI}$$

Let say EI = 30,000 kNm^2

$\boldsymbol{\theta_B}$ **= 0.005 rad** **(ANS)**

Say y_{max} between $0 < x < 6$ as shown in the figure:



$y_{max} = MM' - t_{M/A}$

where $\frac{MM'}{x} = \frac{t_{C/A}}{9} \rightarrow MM' = \frac{x}{3} t_{C/A}$

$$t_{M/A} = +\frac{1}{EI}\left[\frac{1}{2} \cdot x \cdot 20x\left(\frac{1}{3}x\right)\right] = \frac{10x^3}{3EI}$$

$$\therefore\ y_{max} = \left(\frac{x}{9} \cdot \frac{2160}{EI}\right) - \frac{10x^3}{3EI}$$

$$= \frac{1}{EI}\left(240x - \frac{10x^3}{3}\right) \quad \text{.... (3)}$$

Maximum deflection occurs when $\frac{dy}{dx} = 0$

From **Eq. (3)**: $\frac{dy_{max}}{dx} = \frac{1}{EI}\left(240x - \frac{10x^3}{3}\right) = 0$

$x = \sqrt{24} = 4.9\ m\ < 6\ m$ ➔ OK as assumed

$$\therefore\ y_{max} = \frac{1}{EI}\left(240(4.9) - \frac{10(4.9)^3}{3}\right) = \frac{785}{EI}$$

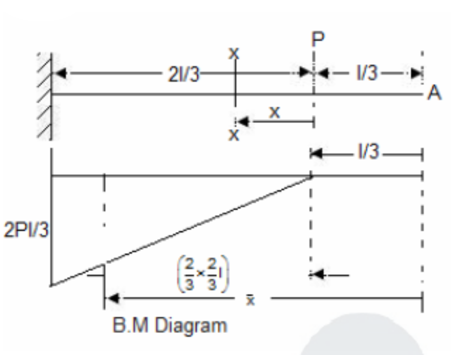
Let say EI = 30,000 kNm^2

$\boldsymbol{y_{max} = \frac{785}{30{,}000} = 26\ mm}\ (\downarrow)$ **(ANS)**

Example:09

A cantilever beam of length ‘l’ is subjected to a concentrated load P at a distance of l/3 from the free end. What is the deflection of the free end of the beam? (EI is the flexural rigidity)

SOLUTION



Moment Area method gives us

$$\delta_A = \frac{\text{Area}}{EI}\bar{x} = \frac{\frac{1}{2}\times\left(\frac{2Pl}{3}\right)\times\left(\frac{2l}{3}\right)\times\left(\frac{l}{3}+\frac{4}{9}l\right)}{EI}$$

$$= \frac{Pl^3}{EI}\times\frac{2}{9}\times\frac{7}{9} = \frac{14}{81}\frac{Pl^3}{EI}$$

$$\text{Alternatively } Y_{max} = \frac{Wa^2}{EI}\left\{\frac{l}{2}-\frac{a}{6}\right\} = \frac{W\left(\frac{2l}{3}\right)^2}{EI}\left\{\frac{l}{2}-\frac{2l/3}{6}\right\}$$

$$= \frac{Wl^3}{EI}\times\frac{4}{9}\times\frac{(9-2)}{18}$$

$$= \frac{14}{81}\frac{Wl^3}{EI}$$

Example:10(choice question)

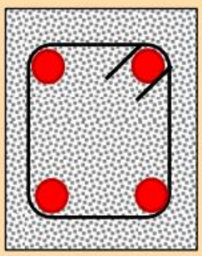
At a certain section at a distance 'x' from one of the supports of a simply supported beam, the intensity of loading, bending moment and shear force arc Wx, Mx and Vx respectively. If the intensity of loading is varying continuously along the length of the beam, then the *invalid* relation is:---(a)

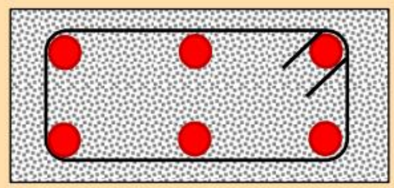
$$(\mathrm{a})\,\mathrm{Slope}\,Q_x = \frac{M_x}{V_x} \quad (\mathrm{b})\,V_x = \frac{dM_x}{dx} \quad (\mathrm{c})\,W_x = \frac{d^2M_x}{dx^2} \quad (\mathrm{d})\,W_x = \frac{dV_x}{dx}$$

3

COLUMN AND STRUTS

CONCRETE COLUMN

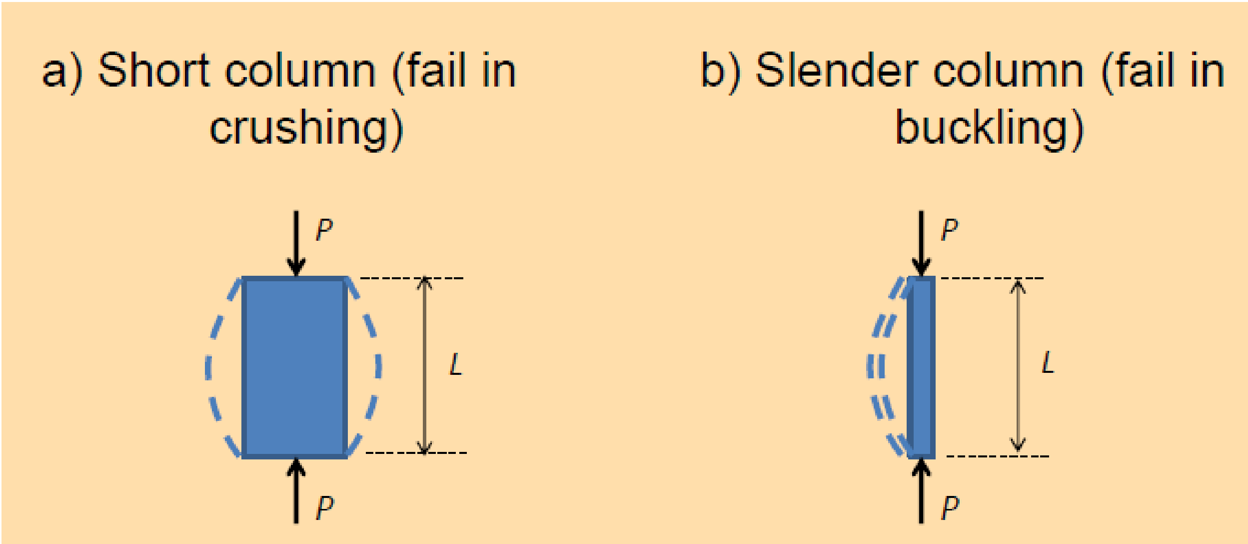




STEEL COLUMN

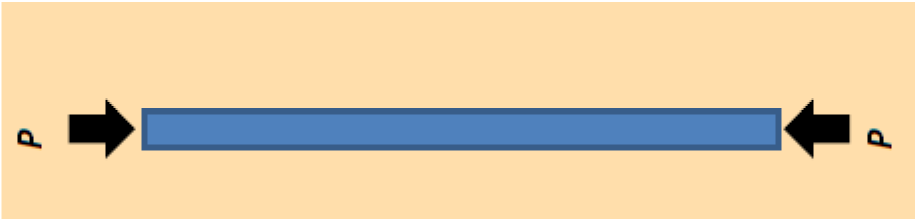
WHAT IS A COLUMN?

- ❖ Columns normally take **compression** force
- ❖ In **vertical** direction; if not, is called **strut**
- ❖ Divided into 2 categories:



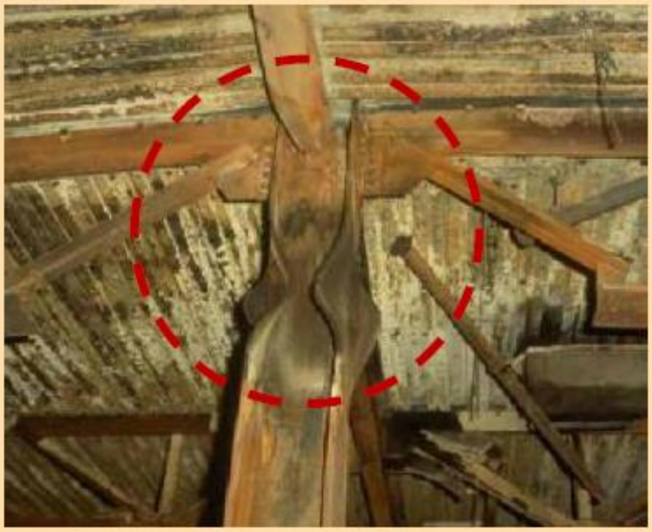
CRITICAL LOAD

- A slender column pinned at both end subjected to load, *P*.
- The maximum axial load that a column can support when it is on the *verge* of buckling is called the critical load, P_{cr}.



(hints, figure should be read vertically)

- Any additional loading will cause the column to buckle and deform laterally.



EULER THEORY (IDEAL COLUMN)

Analysis Assumptions

- The column is **perfectly straight** before loading
- The column is made of **homogeneous material**
- The load is applied through the **centroid** of the cross section
- The material behaves in a **linear-elastic** manner
- The column **buckles and bends** in a single plane

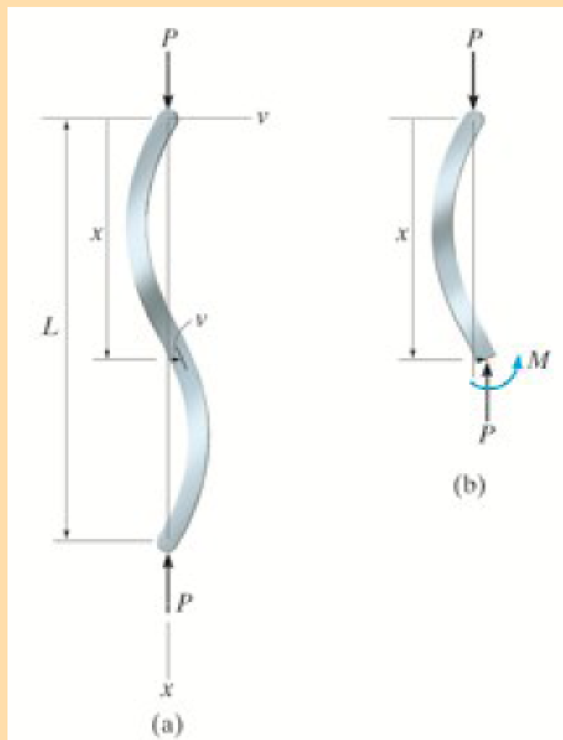
PROBLEMS

By using:

$$EI\frac{d^2v}{dx^2} = -P_{cr}v$$

Prove that:

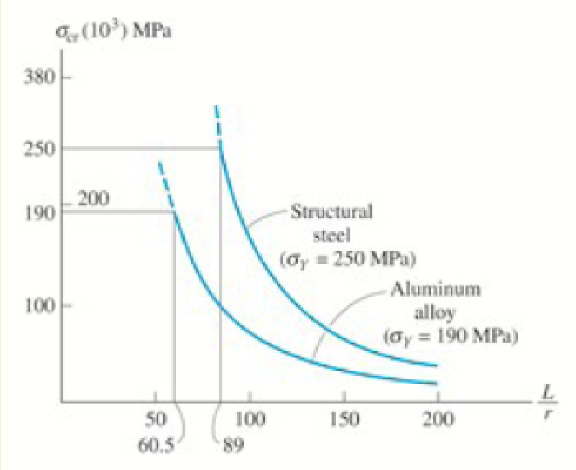
$$P_{cr} = \frac{\pi^2 EI}{L^2}$$



BUCKLING OF COLUMNS

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad \text{or} \quad \sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

P_{cr} = Maximum axial load (kN)
σ_{cr} = Critical stress (N/mm^2)
E = Modulus of Elasticity of the material
I = Least moment of inertia for the column's cross-sectional area
L = Unsupported length of the column
r = Smallest radius of gyration of the column
L/r = Slenderness ratio



IMPORTANT POINTS

- Columns are **long slender** members that are subjected to **axial loads**
- The **critical load** is the maximum axial load that a column can support when it is on the **verge of buckling**. This loading represents a case of neutral equilibrium
- An **ideal column** is initially **perfectly straight**, made of **homogenous** materials, and the load is applied through the **centroid** of the cross section
- A **pin-connected column** will buckle about the principal axis of the cross section having the **least moment of inertia**
- The **slenderness ratio** is L/r, where r is the smallest radius of gyration of the cross section. Buckling will occur about the axis where this ratio gives the greatest value

COLUMNS HAVING VARIOUS TYPES OF SUPPORT

•Euler is used to determine the critical load provided "L" represents the distance between the zero-moment points.

•It is called the column's **effective length, *Le***.

•A dimensionless coefficient, K **(effective-length factor)**, is used to calculate Le. Therefore, $Le=KL$

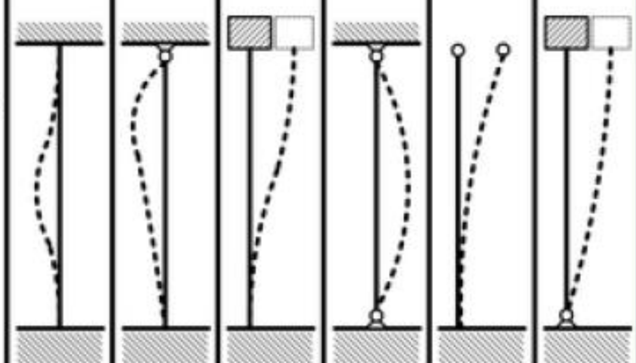
•Thus, we have:

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} \text{ or } \sigma_{cr} = \frac{\pi^2 E}{(KL/r)^2}$$

where $\frac{KL}{r}$ is the effective slenderness ratio

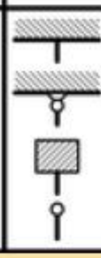
WHAT IS THE VALUE OF *K*?

Buckled shape of column shown by dashed line



Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value K	0.65	0.80	1.2	1.0	2.10	2.0

End condition key



- Rotation fixed and translation fixed
- Rotation free and translation fixed
- Rotation fixed and translation free
- Rotation free and translation free

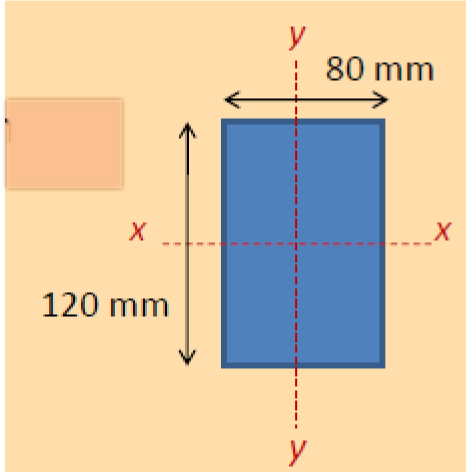
Example:01

A 6 m long concrete column having an Elastic Modulus of 200 kN/mm2 and the cross section shown in the figure is to be used in a building. Determine the maximum allowable axial load (P_{allow}) the column can support so that it does not buckle. The safety factor is taken as 2.0. Given that:

a)Both end of the column are pinned ends

b)Both end of the column are fixed ends

c)One end is fixed and the other end is free to move



Determine the second moment area of the column:

$$I_x = \frac{bh^3}{12} = \frac{80\times120^3}{12} = 11.52\ \times 10^6\ \text{mm}^4$$

$$I_y = \frac{bh^3}{12} = \frac{120\times80^3}{12} = 5.12\ \times 10^6\ \text{mm4}$$

∴ The column will buckle in the y-y axis (smallest I)

a) Both end of the column are **pinned** ends

$$L_e = 1.0L$$

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2 \times 200 \times 5.12 \times 10^6}{(1.0 \times 6000)^2} = 280.7\ kN$$

∴ Maximum allowable axial load, P_{allow}:

$$P_{allow} = \frac{P_{cr}}{F.O.S} = \frac{280.7}{2.0} = \mathbf{140.35\ kN}\ \textbf{(ANS)}$$

b) Both end of the column are **fixed** ends

$$L_e = 0.5L$$

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2 \times 200 \times 5.12 \times 10^6}{(0.5 \times 6000)^2} = 1122.94\ kN$$

∴ Maximum allowable axial load, P_{allow}:

$$P_{allow} = \frac{P_{cr}}{F.O.S} = \frac{1122.94}{2.0} = \mathbf{561.47 kN} \qquad \textbf{(ANS)}$$

c) One end is **fixed** and the other one is **free to move**

$$L_e = 2.0L$$

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2 \times 200 \times 5.12 \times 10^6}{(2.0 \times 6000)^2} = 70.20\ kN$$

∴ Maximum allowable axial load, P_{allow}:

$$P_{allow} = \frac{P_{cr}}{F.O.S} = \frac{70.20}{2.0} = \mathbf{35.10 kN} \qquad \textbf{(ANS)}$$

Example:02

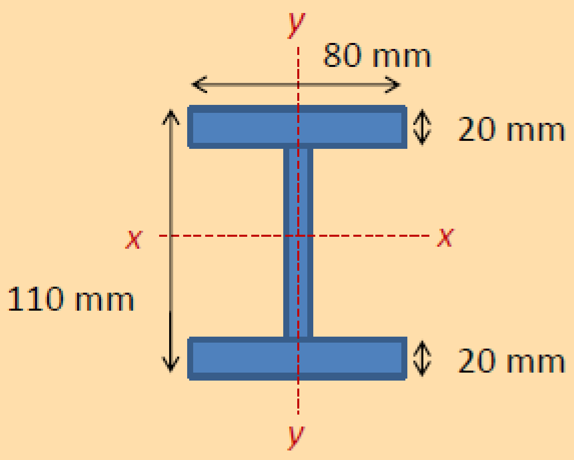
A 6 m long steel column having an Elastic Modulus of 200 kN/mm^2 and the cross section shown in the figure is to be used in a building. The safety factor is taken as 2.0.

a)Determine the slenderness ratio of the column

b)Determine the critical load (P_{cr})

c)Determine the maximum allowable axial load (P_{allow})

Assumed both ends of the column are pinned.



Determine the second moment of area, $I = \frac{bh^3}{12} + A\bar{y}^2$

$$I_x = \left[\frac{80\times20^3}{12} + (80 \times 20 \times 55^2)\right] \times 2 + \frac{10\times90^3}{12} = 10.394 \times 10^6 \text{ mm}^4$$

$$I_y = \left(\frac{20\times80^3}{12}\right) \times 2 + \frac{90\times10^3}{12} = 1.714 \times 10^6 \text{ mm}^4$$

$\therefore$ ***The column will buckle in the y-y axis (smallest I)***

a) Determine the slenderness ratio

$$A = (80 \times 20)2 + (90 \times 10) = 4100 \text{ mm}^2$$

Radius of gyration, $r = \sqrt{\frac{I}{A}} = \sqrt{\frac{1.714\times10^6}{4100}} = 20.45 \text{ mm}$

$\therefore$ The slenderness ratio, $\frac{L}{r} = \frac{6000}{20.45} = 293.45$

b) Determine the critical load (P_{cr})

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2 \times 200 \times 1.714 \times 10^6}{(1.0 \times 6000)^2} = 94\ kN$$

c) Determine the maximum allowable axial load (P_{allow})

$$P_{allow} = \frac{P_{cr}}{F.O.S} = \frac{94}{2.0} = 47kN$$

Rankine's Formula

We have learnt that Euler's formula gives correct results only for very long columns. But what happens when the column is a short or the column is not very long. On the basis of results of experiments performed by Rankine, he established an empirical formula which is applicable to all columns whether they are short or long. The empirical formula given by Rankine is known as Rankine's formula, which is given as

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_E}$$

$$P_R \ = Crippling\ load\ by\ Rankine's\ Formula$$

$$P_C = f_C . A$$

$$= Ultimate\ crushing\ load\ for\ the\ column$$

$$P_E = \frac{\pi^2 EI}{{l_e}^2}$$

$$= Crippling\ load\ \ by\ Euler's\ formula$$

$$P_R = \frac{f_C . A}{1 + a\left(\frac{l_e}{r}\right)^2} = \frac{P_C}{1 + a\left(\frac{l_e}{r}\right)^2}$$

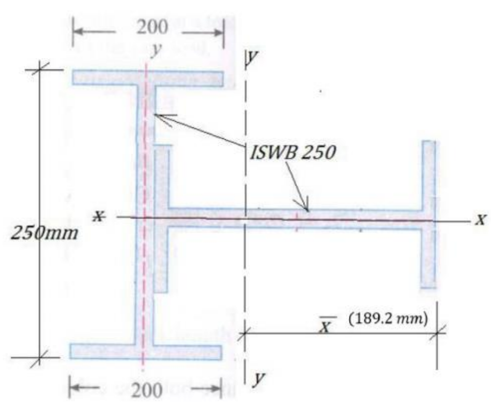
P_C = *Crushing load of the column material*
f_C = *Crushing stress of the column material*
A = *Cross − sectional area of the column*
a = *Rankine's constant* $\left(= \frac{f_C}{E\,\pi^2}\right)$
L_e = *Equivalent length of the column,*
r= *Least radius of gyration*

Rankine's constant for various materials

S.No	Material	f_C in MPa	$a = \frac{f_C}{\pi^2 E}$
1	Mild steel	320	$\frac{1}{7500}$
2	Cast iron	550	$\frac{1}{1600}$
3	Wrought iron	250	$\frac{1}{9000}$
4	timber	40	$\frac{1}{750}$

QUIZ 1

A column of 9 m long has a cross section shown in figure. The column is pinned at both ends. If the column is subjected to an axial load equal in value ¼ of the Euler's critical load for the column. Determine the factor of safety on the Rankine's ultimate stress value. Take fc = 326 N/mm^2 , Rankine's constant a = 1 7500 , E =200 Gpa. Properties of one RSJ area = 5205 mm^2 . I_{xx}=5943.1× 104 $cm4$, I_{yy}=857.5× 104 $cm4$, Thickness of the web=6.7 mm.



QUIZ 2

A hollow cylindrical cast iron column is 4 m long, both ends being fixed. Design the column to carry an axial load of 250 kN. Use Rankine's formula and adopt a factor of safety of 5. The internal diameter may be taken as 0.8 times the external diameter. Take fc = 550 N/mm^2 and a = 1 /1600

QUIZ 3

A short length of tube having Internal diameter and external diameter are 4 cm and 5 cm respectively, which failed in compression at a load of 250 kN. When a 1.8 m length of the same tube was tested as a strut with fixed ends, the load failure was 160 kN. Assuming that σ_c in Rankin's formula is given by the first test, find the value of the constant α in the same formula. What will be the crippling load of this tube if it is used as a strut 2.8 m long with one end fixed and the other hinged?

Example:03(discussion)

A hollow mild steel tube 6 m long 4 cm internal diameter and 6 mm thick is used as a strut with both ends hinged. Find the crippling load and safe load taking factor of safety as 3.
Take E = 2 x 10^5 N / mm^2.

Example:04(discusion)

A simply supported beam of length 4m is subjected to a uniformly distributed load of 30 KN/m over the whole span and deflects 15 mm at the centre. Determine the crippling load the beam is used as a column with the following conditions: (i) One end fixed and another end hinged (ii) Both the ends pin jointed.

Example:05(discussion)

A hollow cylindrical cast iron column is 4 m long with both ends fixed. Determine the minimum diameter of the column if it has to carry a safe load of 250 KN with a factor of safety of 5. Take the internal diameter as 0.8 times the external diameter. Take σ_C = 550 N /mm2 and. a = 1/1600 in Rankine's formul

Example:06

Compare the ratio of the strength of solid steel column to that of the hollow steel column of the same cross-sectional area. The internal diameter of the hollow column is 3/4th of the external diameter. The columns have the same length and are pinned at both ends. Use Euler's theory.

<u>SOLUTION</u>

Let '*d*' be the diameter of the solid strut and *D* be the outer diameter of the hollow strut. As the cross-sectional areas are same,

$$\frac{\pi}{4} d^2 = \frac{\pi}{4}\left[D^2 - \left(\frac{3D}{4}\right)^2\right]$$

$$d^2 = D^2 - \frac{9}{16} D^2 = \frac{7}{16} D^2$$

Let P_{crh} = buckling load of the hollow column

P_{crs} = buckling load of solid column

I_h = least moment of inertia of hollow column

$$= \frac{\pi}{64}\left[D^4 - \left(\frac{3D}{4}\right)^4\right] = \frac{\pi}{64} \times \frac{175}{256} D^4$$

I_s = Least moment of the solid column

$$= \frac{\pi}{64} d^4$$

Now $P_{crh} = \dfrac{\pi^2 EI_h}{L^2}$ and $P_{crs} = \dfrac{\pi^2 EI_s}{L^2}$

$$\therefore \quad \frac{P_{crh}}{P_{crs}} = \frac{I_h}{I_s} = \frac{\frac{\pi}{64} \times \frac{175\, D^4}{256}}{\frac{\pi}{64} \times d^4}$$

$$= \frac{175\, D^4}{256\, d^4} = \frac{175\, D^4}{256\left(\frac{7}{16} D^2\right)^2} = \frac{25}{7}$$

Thus $\dfrac{P_{crh}}{P_{crs}}$ = **3.571**. **Ans.**

Example:07

A solid round bar of 60 mm diameter and 2.5 m long is used as a strut. Find the safe compressive load for the strut using Euler's formula if (*a*) both ends are hinged (*b*) both ends are fixed. Take $E = 2 \times 10^5$ N/mm^2 and factor of safety = 3.

SOLUTION

l = 2.5 m = 2500 mm

d = 60 mm, $E = 2 \times 10^5$ N/mm^2

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} \times 60^4 = 636172.51 \text{ mm}^4$$

Factor of safety = 3.

(a) Both ends are hinged:

$$P_{cr} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2 \times 10^5 \times 636172.51}{(2500)^2} = 200920 \text{ N}$$

$$= 200.92 \text{ kN}$$

$$\therefore \quad \text{Safe load} = \frac{P_{cr}}{\text{Factor of safety}} = \frac{200.92}{3} = \mathbf{66.97\ kN}.$$

(b) Both ends are fixed:

$$P_{cr} = \frac{4\pi^2 EI}{l^2} = \frac{4 \times \pi^2 \times 2 \times 10^5 \times 636172.51}{(2500)^2}$$

$$= 803682 \text{ N} = 803.682 \text{ kN}$$

$$\therefore \quad \text{Safe load} = \frac{P_{cr}}{\text{Factor of safety}} = \frac{803.682}{3} = \mathbf{262.89\ kN}. \qquad \textbf{Ans.}$$

Example:08

What is the ratio of the strength of a solid steel column of 150 mm diameter to that of a hollow circular steel column of the same cross-sectional area and a wall thickness of 15 mm? The two columns have the same length and similar end conditions.

<u>SOLUTION</u>

Diameter of circular column d = 150 mm

$$\therefore \quad \text{C.S. Area} = \frac{\pi}{4} \times 150^2$$

Let the thickness of circular hollow column be t = 15 mm

Let external diameter of hollow circular column be D = mm

$\therefore$ Its internal diameter = $D - 2t = D - 2 \times 15 = (D - 30)$ mm

$$\therefore \quad \text{C.S. area} = \frac{\pi}{4} \{D^2 - (D-30)^2\}$$

This area is same as that of solid column

$$\therefore \quad \frac{\pi}{4}\{D^2 - (D-30)^2\} = \frac{\pi}{4} \times 150^2$$

$$D^2 - \{D^2 - 60D + 900\} = 150^2$$

$$60D = 22500 + 900 = 23400$$

$$\therefore \quad D = 390 \text{ mm}$$

$\therefore$ Internal diameter of hollow column = 390 – 30 = 360 mm.

Least moment of inertia:

$$I_s = \frac{\pi}{64} \times 150^4 = 24850488.7 \text{ mm}^4$$

$$I_h = \frac{\pi}{64}(390^4 - 360^4) = 311128119.5 \text{ mm}^4$$

$$P_{crh} = \frac{\pi^2 E I_h}{l_e^2}$$

$$P_{crs} = \frac{\pi^2 E I_s}{l_e^2}$$

$$\therefore \frac{P_{crh}}{P_{crs}} = \frac{I_h}{I_s} = \frac{311128119.5}{24850488.7} = \mathbf{12.52}. \quad \textbf{Ans.}$$

Example:09

Find the Euler's crushing load for a hollow cylindrical cast iron column 120 mm external diameter and 20 mm thick, if it is 4.2 m long and is hinged at both ends. Take *E* = 80 kN/mm2. Compare this load with the crushing load as given by Rankine's formula using constants *fc* = 550 N/mm2 and *a* = 1/1600. For what length of strut does the Euler's formula cease to apply?

<u>SOLUTION</u>

External diameter = 120 mm

Thickness = 20 mm

Internal diameter = 120 – 2 × 20 = 80 mm

$$\text{Least moment of inertia} = \frac{\pi}{64}(120^4 - 80^4)$$

$$= 8168140.89 \text{ mm}^4$$

Column is hinged at both ends.

$$\therefore \quad l_e = 4.2 \text{ m} = 4200 \text{ mm}$$

$$\text{Euler's buckling load} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 80 \times 10^3 \times 8168140.89}{(4200)^2}$$

$$= \mathbf{365606.89 \text{ N}}. \quad \textbf{Ans.}$$

$$A = \frac{\pi}{4}(120^2 - 80^2) = 6283.18 \text{ mm}^2$$

$$K^2 = \frac{I}{A} = \frac{8168140.89}{6283.18} = 1300$$

$\therefore \qquad K^2 = 36.05 \text{ mm}$

$$\therefore \quad \text{Rankine's critical load } P_R = \frac{f_c A}{1 + a\left(\frac{l}{K}\right)^2} = \frac{550 \times 6283.18}{1 + \frac{1}{1600}\left(\frac{4200}{36.05}\right)^2} = 364415.16 \text{ N}$$

$$\therefore \quad \frac{P_E}{P_R} = \frac{365606.89}{364415.16} = \mathbf{1.003}. \qquad \textbf{Ans.}$$

Now, $$P_E = \frac{\pi^2 EI}{l^2}$$

Equating it to crushing load, we get

$$\frac{\pi^2 EI}{l^2} = f_c A$$

$$\frac{\pi^2 E}{l^2} K^2 = f_c$$

$$\frac{\pi^2 \times 80 \times 10^3 \times 1300}{l^2} = 550$$

$$l^2 = \frac{\pi^2 \times 80 \times 1000 \times 1300}{550}$$

$$l = \mathbf{1366.108 \text{ mm}}. \qquad \textbf{Ans.}$$

Example:10

An ISLB 300 section is provided with a flange plate 200 mm × 12 mm for each flange. The composite member is used as a column with one end fixed and the other end hinged. Calculate the length of the column for which, crippling loads given by Rankine's formula and Euler's formula will be the same. Take E = 210 kN/mm2, fc = 330 N/mm2, a = 1/7500

Properties of ISLB 300 section are:

Overall width = 150 mm, Overall depth = 300 mm,Thickness of flange = 9.4 mm, Thickness of web = 6.7 mm ,I_{xx} = 73.329 × 106 mm^4,I_{yy} = 3.762 × 106 mm^4 ,A = 4808 mm^2

Solution: $f_c = 330$ N/mm², $a = \dfrac{1}{7500}$, $E = 210 \times 10^3$ N/mm²

$$\text{Area } A = 4808 \text{ mm}^2$$

$$I_{xx} = 73.329 \times 10^6 \text{ mm}^4, \quad I_{yy} = 3.762 \times 10^6 \text{ mm}^4$$

Sectional area of ISLB 300 column.

$$A = 4808 + 2 \times (200 \times 12) = 9608 \text{ mm}^2$$

Moment of inertia about $x - x$ axis.

$$I_{xx} = 73.329 \times 10^6 + 2\left[\frac{200 \times 12^3}{12} + (200 \times 12)\, 156^2\right]$$

$$= 190199406 \text{ mm}^4$$

Moment of inertia about y-y axis.

$$I_{yy} = 3.762 \times 10^6 + 2\left(\frac{12 \times 200^3}{12}\right) = 19762000 \text{ mm}^4$$

Since $I_{yy} < I_{xx}$, the column buckles about y-y axis.

$$\therefore \quad I = I_{\min} = 19762000 \text{ mm}^4$$

$$\text{Least radius of gyration} = K = \sqrt{\frac{I}{A}} = \sqrt{\frac{19762000}{9608}}$$

$$= 45.35 \text{ mm.}$$

Let L = effective length

$$\frac{\pi^2 EI}{L^2} = \frac{f_c A}{1 + a\dfrac{L^2}{K^2}}$$

$$\frac{\pi^2 \times 210 \times 10^3 \times 19762000}{L^2} = \frac{330 \times 9608}{1 + \dfrac{1}{7500}\dfrac{L^2}{2056.82}}$$

$$12918229.65 = \frac{L^2}{1 + L^2 \times 6.4825 \times 10^{-8}}$$

$$12918229.65 + 0.8374\, L^2 = L^2$$

$$\therefore \quad L = 8914 \text{ mm}$$

Therefore, required actual length for one end hinged and other end fixed column for which critical load by Rankine's formula and Euler's formula will be equal is

$$l = \sqrt{2}\, L = \sqrt{2} \times 8914 = 12606 \text{ mm}$$

$$= \mathbf{12.606 \text{ m.}}$$

Ans.

Example:11

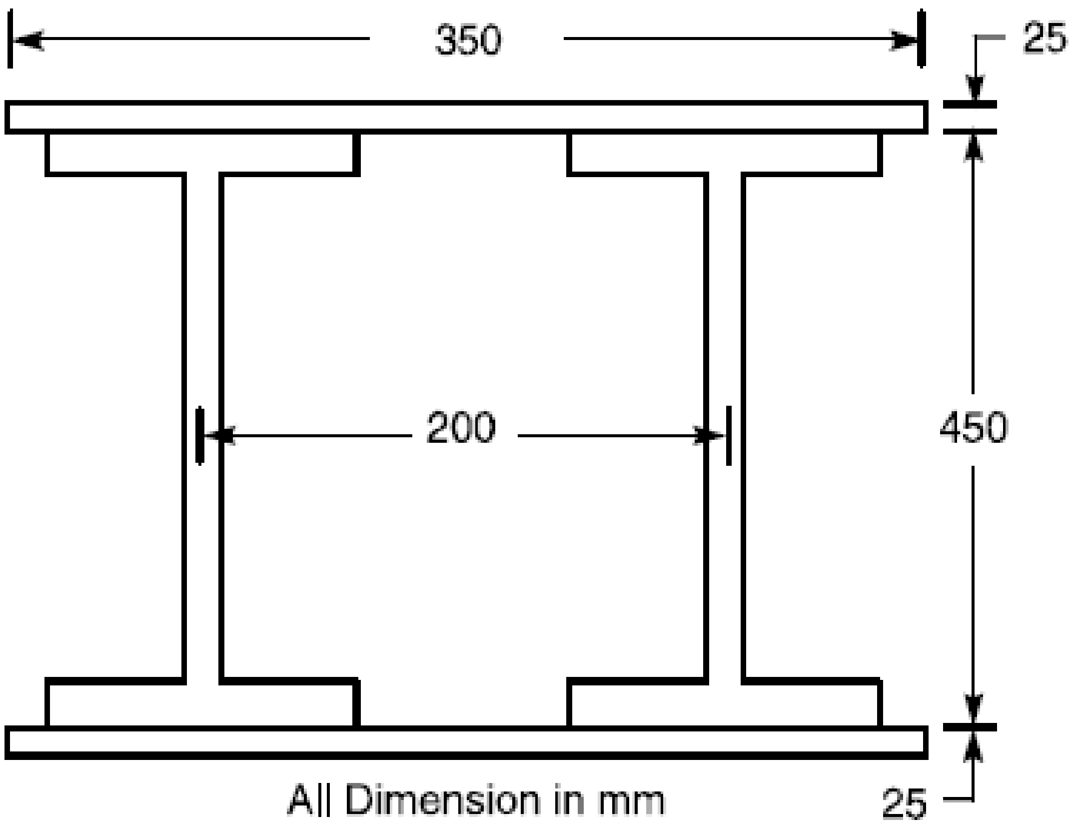
A built up steel column, 8 m long and ends firmly fixed is having cross-section as shown in Figure below. The properties of I-section are Area = 9300 mm^2, I_{xx} = 3 × 106 min4, I_{yy} = 8.4 × 106 mm4. Determine:

(*i*) The safe axial load the column can carry with a factor of safety of 3.5 using (*a*) Euler's Formula, (*b*) Rankine's Formula.

(*ii*) The length of the column for which both formulae give the same crippling load.

(*iii*) The length of the column for which the Euler's formula ceases to apply.

Take $E = 2 \times 10^5$ N/mm^2, fc = 330 N/mm^2, a = 1/7500



Solution: (*i*) Length of the column = 8 m = 8000 mm

$$\text{Factor of safety} = 3.5, \quad f_c = 330 \text{ N/mm}^2,$$

$$a = \frac{1}{7500}, \quad E = 2 \times 10^5 \text{ N/mm}^2$$

$$A = 2\,(9300 + 350 \times 25) = 36100 \text{ mm}^2$$

Moment of inertia of column section about *x-x* axis:

$$I_{xx} = 2 \times 3 \times 10^6 + 2\left[\frac{350 \times 25^3}{12} + 350 \times 25 \times 237.5^2\right]$$

$$= 994020833.5 \text{ mm}^4$$

Moment of inertia of the column section about *y-y* axis:

$$I_{yy} = 2 \times 8.4 \times 10^6 + 2\left[\frac{25 \times 350^3}{12} + 9300 \times 100^2\right]$$

$$= 381445833.3 \text{ mm}^4$$

$$I_{yy} < I_{xx}$$

$$\therefore \quad I = I_{min} = 381445833.3 \text{ mm}^4$$

$$K = \sqrt{\frac{I}{A}} = \sqrt{\frac{381445833.3}{36100}}$$

$$= 102.793$$

Since column is fixed at both ends,

$$P_E = \frac{4\pi^2 EI}{l^2} = \frac{4 \times \pi^2 \times 2 \times 10^5 \times 381445833.3}{(8000)^2}$$

$$= 47058993.44 \text{ N}$$

$$\therefore \quad \text{Safe Load } P = \frac{P_E}{3.5} = \frac{47058993.44}{3.5} = 13445.40 \times 10^3 \text{ N}$$

$$= \mathbf{13445.4\ kN.} \qquad \textbf{Ans.}$$

(*b*)

$$P_R = \frac{f_c A}{1 + a\left(\dfrac{L}{K}\right)^2} \qquad \text{where } L = \frac{l}{2} = \frac{8000}{2} = 4000 \text{ mm}$$

$$= \frac{330 \times 36100}{1 + \dfrac{1}{7500}\left(\dfrac{4000}{102.793}\right)^2}$$

$$= 9911.806 \times 10^3 \text{ N}$$

$$= 9911.806 \text{ kN.}$$

$$\therefore \quad \text{Safe load } P = \frac{9911.806}{3.5} = \mathbf{2831.95\ kN.} \qquad \textbf{Ans.}$$

(ii) Let L_1 be the effective length

$$\frac{\pi^2 EI}{L_1^2} = \frac{f_c A}{1 + a\dfrac{L_1^2}{K^2}}$$

$$\frac{\pi^2 \times 2 \times 10^5 \times 381445833.3}{L_1^2} = \frac{330 \times 36100}{1 + \dfrac{1}{7500}\dfrac{L_1^2}{(102.793)^2}}$$

or $$0.797\, L_1^2 + 63203550.35 = L_1^2$$

or $$L_1 = 17670 \text{ mm}$$

$$= \mathbf{17.67 \ m.}$$ **Ans.**

(iii) Let 'l' be the length of column for which Euler's formula ceases to apply. Then

$$P_E = \frac{4\,\pi^2 EI}{l^2}$$

$$f_c A = \frac{4\,\pi^2 EI}{l^2}$$

$$330 = \frac{4\,\pi^2 EK^2}{l^2} = \frac{4\,\pi^2 \times 2 \times 10^5 \times (102.793)^2}{l^2}$$

$\therefore$ $$l^2 = 252815021.4$$

$$l = 15900 \text{ mm}$$

$$= \mathbf{15.9 \ m.}$$ **Ans.**

Example:12

A hollow circular column of internal diameter 20 mm and external diameter 40 mm has a total length of 5m. One end of the column is fixed and the other end is hinged. Find out the crippling stress of the column if E = 2 × 105 N/mm^2 . Also findout the shortest length of this column for which Euler's formula is valid taking the yield stress equal to 250 N/mm^2

Solution.

d=20 mm; D=40 mm; $l = 5\ mm$; E=2×10^5N/mm^2.

Euler's crippling load for one end fixed and the

Hinged, $P = \frac{2\pi^2 EI}{l^2}$

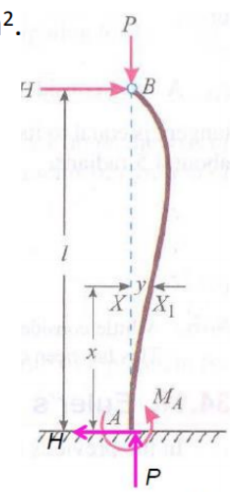
Euler's crippling stress, $p_c = \frac{P_c}{A} = \frac{2\pi^2 EI}{Al^2}$

Area of the column, $A = \frac{\pi(D^2 - d^2)}{4} = \frac{\pi(40^2 - 20^2)}{4}$

$= 942.48\ mm^2$

Moment of inertia, $I = \frac{\pi(D^4 - d^4)}{64} = \frac{\pi(40^4 - 20^4)}{64}$

$= 117809.75\ mm^4$



Euler's crippling stress, $p_c = \frac{P_c}{A} = \frac{2\pi^2 EI}{Al^2} = \frac{2\pi^2 \times 2 \times 10^5 \times 117809.73}{942.48 \times 5000^2}$

$$= 19.74 \text{ N}/mm^2$$

Yield stress= 250 N/mm².

$$\frac{l}{k} = \sqrt{\frac{2\pi^2 E}{250}} = \sqrt{\frac{2\pi^2 \times 2 \times 10^5}{250I}} = 125.66$$

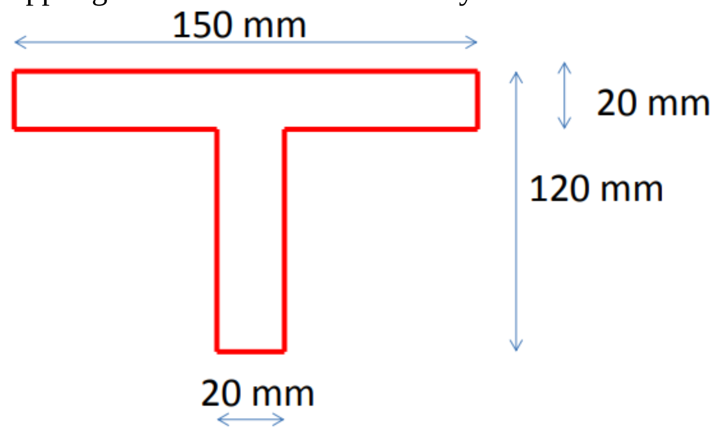
$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{117809.73}{942.48}} = 11.18$$

$$l = 125.66 \times k = 125.66 \times 11.18 = 1404.9\ mm$$

∴Shortest length of this column= 1.4 m.

Example:13

A T-section 150 mm x 120 mm x 20 mm is used as a strut of 4 m long with hinged at its both ends. Calculate the crippling load if modulus of elasticity for the material be 2.0 x 10^5 N/mm^2 .

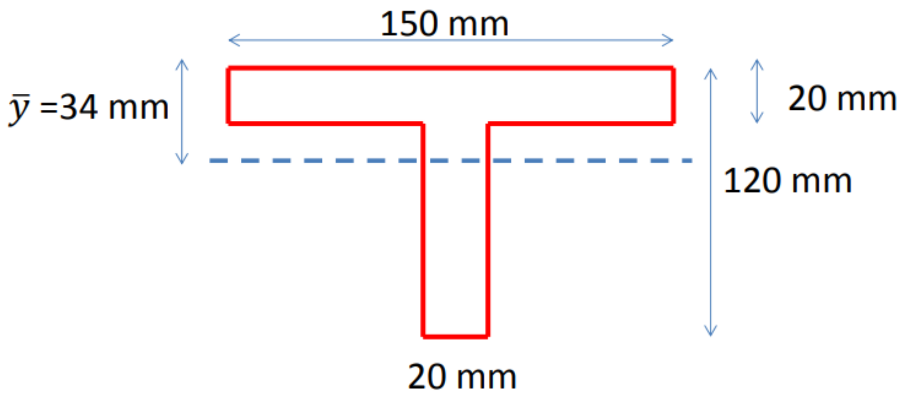


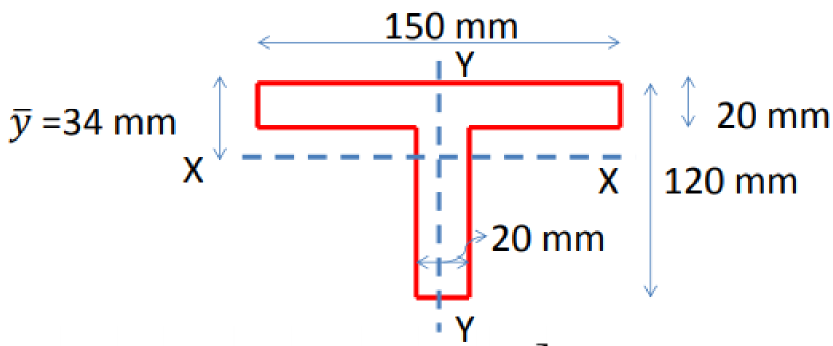
First of all, let us find out the C.G of the section. Let $\bar{y}$ be the distance between C.G of the section from top of the flange

By geometry of the figure, $\bar{x} = 0$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$= \frac{(150 \times 20) \times 10 + (100 \times 20) \times 70}{(150 \times 20) + (100 \times 20)} = 34\ mm$$





$$I_{XX} = \left[\frac{1}{12} \times 150 \times 20^3 + (150 \times 20) \times 24^2\right] + \left[\frac{20 \times 100^3}{12} + 2000 \times 36^2\right]$$

$$= 6.0867 \times 10^6 mm^4$$

And

$$I_{YY} = \frac{20 \times 150^3}{12} + \frac{100 \times 20^3}{12}$$

$$= 5.6917 \times 10^6 mm$$

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Since I_{YY} is less

∴ The colun y direction

Given End condition : both ends hinged

∴ $l_e = l$

∴ $P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2.0 \times 10^5 \times 5.6917 \times 10^6}{4000^2}$

$= 702185\ N$

THE END